

# ANOSOV DIFFEOMORPHISMS ON NILMANIFOLDS UP TO DIMENSION 8

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**ABSTRACT.** After more than thirty years, the only known examples of Anosov diffeomorphisms are hyperbolic automorphisms of infranilmanifolds. It is also important to note that the existence of an Anosov automorphism is a really strong condition on an infranilmanifold. Any Anosov automorphism determines an automorphism of the rational Lie algebra determined by the lattice, which is hyperbolic and unimodular (and conversely ...). These two conditions together are strong enough to make of such rational nilpotent Lie algebras (called Anosov Lie algebras) very distinguished objects. In this paper, we classify Anosov Lie algebras of dimension less or equal than 8.

As a corollary, we obtain that if an infranilmanifold of dimension  $n \leq 8$  admits an Anosov diffeomorphism  $f$  and it is not a torus or a compact flat manifold (i.e. covered by a torus), then  $n=6$  or  $8$  and the signature of  $f$  necessarily equals  $\{3, 3\}$  or  $\{4, 4\}$ , respectively. We had to study the set of all rational forms up to isomorphism of many real Lie algebras, which is a subject on its own and it is treated in a section completely independent of the rest of the paper.

## 1. INTRODUCTION

A diffeomorphism  $f$  of a compact differentiable manifold  $M$  is called *Anosov* if it has a global hyperbolic behavior, i.e. the tangent bundle  $TM$  admits a continuous invariant splitting  $TM = E^+ \oplus E^-$  such that  $df$  expands  $E^+$  and contracts  $E^-$  exponentially. These diffeomorphisms define very special dynamical systems and it is then a natural problem to understand which are the manifolds supporting them (see [29]). After more than thirty years, the only known examples are hyperbolic automorphisms of infranilmanifolds (called *Anosov automorphisms*) and it is conjectured that any Anosov diffeomorphism is topologically conjugate to one of these (see [23]). The conjecture is known to be true in many particular cases: J. Franks [10] and A. Manning [22] proved it for Anosov diffeomorphisms on infranilmanifolds themselves; Y. Benoist and F. Labourie [2] in the case the distributions  $E^+, E^-$  are differentiable and the Anosov diffeomorphism preserves an affine connection (for instance a symplectic form); and J. Franks [10] when  $\dim E^+ = 1$  (see also [12] for expanding maps). Since Anosov automorphisms have many additional dynamical properties (see [31]), a general resolution of the conjecture would be of great relevance.

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It is also important to note that the existence of an Anosov automorphism is a really strong condition on an infranilmanifold. An infranilmanifold is a quotient  $N/\Gamma$ , where  $N$  is a nilpotent Lie group and  $\Gamma \subset K \ltimes N$  is a lattice (i.e. a discrete cocompact subgroup) which is torsion-free and  $K$  is a compact subgroup of  $\text{Aut}(N)$ . Among some other more technical obstructions (see [21] for further information), the first natural obstruction for the infranilmanifold  $N/\Gamma$  to admit an Anosov automorphism is that the nilmanifold  $N/(\Gamma \cap N)$ , which is a finite cover of  $N/\Gamma$ , has to do so.

In the case of a nilmanifold  $N/\Gamma$  (i.e. when  $\Gamma \subset N$ ), any Anosov automorphism determines an automorphism  $A$  of the rational Lie algebra  $\mathfrak{n}^{\mathbb{Q}} = \Gamma \otimes \mathbb{Q}$ , the Lie algebra of the rational Mal'cev completion of  $\Gamma$ , which is *hyperbolic* (i.e.  $|\lambda| \neq 1$  for any eigenvalue  $\lambda$  of  $A$ ) and *unimodular* (i.e.  $[A]_{\beta} \in \text{GL}_n(\mathbb{Z})$  for some basis  $\beta$  of  $\mathfrak{n}^{\mathbb{Q}}$ ). Recall that  $\mathfrak{n}^{\mathbb{Q}}$  is a rational form of the Lie algebra  $\mathfrak{n}$  of  $N$ . These two conditions together are strong enough to make of such rational nilpotent Lie algebras (called *Anosov Lie algebras*) very distinguished objects. It is proved in [15] and [5] that if  $\Gamma_1$  and  $\Gamma_2$  are commensurable (i.e.  $\Gamma_1 \otimes \mathbb{Q} \simeq \Gamma_2 \otimes \mathbb{Q}$ ) then  $N/\Gamma_1$  admits an Anosov automorphism if and only if  $N/\Gamma_2$  does. All this suggests that the class of rational Anosov Lie algebras is the key algebraic structure to study if one attempts to classify infranilmanifolds admitting an Anosov diffeomorphism.

Finally, if one is interested in just those Lie groups which are simply connected covers of such infranilmanifolds, then the objects to be studied are real nilpotent Lie algebras  $\mathfrak{n}$  supporting a hyperbolic automorphism  $A$  such that  $[A]_{\beta} \in \text{GL}_n(\mathbb{Z})$  for some  $\mathbb{Z}$ -basis  $\beta$  of  $\mathfrak{n}$  (i.e. with integer structure constants). Such Lie algebras will also be called *Anosov*. We note that a real Lie algebra is Anosov if and only if it has an Anosov rational form.

The following would be then a natural program to classify all the infranilmanifolds up to homeomorphism of a given dimension  $n$  which admits an Anosov diffeomorphism:

- (i) Find all  $n$ -dimensional Anosov Lie algebras over  $\mathbb{R}$ .
- (ii) For each real Lie algebra  $\mathfrak{n}$  obtained in (i), determine which rational forms of  $\mathfrak{n}$  are Anosov.
- (iii) For each rational Lie algebra  $\mathfrak{n}^{\mathbb{Q}}$  from (ii), classify up to isomorphism all the lattices  $\Gamma$  in  $N$ , the nilpotent Lie group with Lie algebra  $\mathfrak{n}^{\mathbb{Q}} \otimes \mathbb{R}$ , such that  $\Gamma \otimes \mathbb{Q} = \mathfrak{n}^{\mathbb{Q}}$ . In other words, classify up to isomorphism all the lattices in the commensurability class corresponding to  $\mathfrak{n}^{\mathbb{Q}}$ .
- (iv) Given a nilmanifold  $N/\Gamma$  from (iii), decide which of the finitely many infranilmanifolds  $N/\Lambda$  essentially covered by  $N/\Gamma$  (i.e.  $\Lambda \cap N \simeq \Gamma$ ) admits an Anosov automorphism, that is, a hyperbolic automorphism  $\varphi$  of  $N$  such that  $\varphi(\Lambda) = \Lambda$  (see [21]).

Parts (i) and (ii) have been solved for dimension  $n \leq 6$  in [3] and [20], yielding only two algebras over  $\mathbb{R}$ :  $\mathfrak{h}_3 \oplus \mathfrak{h}_3$  and  $\mathfrak{f}_3$  (see Table 1). There are some other families of real Anosov Lie algebras in the literature (see Remark 3.8). Besides these examples in somewhat sporadic dimensions, there is a construction in [18] proving that  $\mathfrak{n} \oplus \mathfrak{n}$  is Anosov for any real graded nilpotent Lie algebra  $\mathfrak{n}$  which admits at least one rational form  $\mathfrak{n}^{\mathbb{Q}}$ . We note that the existing Anosov rational form is not necessarily  $\mathfrak{n}^{\mathbb{Q}} \oplus \mathfrak{n}^{\mathbb{Q}}$ . Since for instance any 2-step nilpotent Lie algebra is graded, this construction shows that part (i) of the program above is already a wild problem for  $n$  large. Furthermore, by using the classification of nilpotent Lie

algebras in low dimensions (see [19, 27]), we can assert that there are at least 18 real Anosov Lie algebras of dimension 10, 68 in dimension 12 and more than 100, together with some curves in dimension 14.

In view of this fact, the aim of this paper is to approach the classification in small dimensions. We have classified up to isomorphism real and rational Lie algebras of dimension  $\leq 8$  which are Anosov. In other words, we have solved parts (i) and (ii) of the program for  $n = 7$  and  $n = 8$ . We refer to Tables 1 and 3 for a quick look at the results obtained. Without an abelian factor, there are only three 8-dimensional real Lie algebras which are Anosov and none in dimension 7. This is a really small list, bearing in mind that there exist several one and two-parameters families and hundreds of isolated examples of 7 and 8-dimensional nilpotent Lie algebras, and there is not even a full classification in dimension 8.

One of the corollaries which might be interesting from a dynamical point of view is that if an infranilmanifold of dimension  $n \leq 8$  admits an Anosov diffeomorphism  $f$  and it is not a torus or a compact flat manifold (i.e. covered by a torus), then  $n = 6$  or  $8$  and the signature of  $f$ , defined by  $\{\dim E^+, \dim E^-\}$ , necessarily equals  $\{3, 3\}$  or  $\{4, 4\}$ , respectively.

We now give an idea of the structure of the proof. The *type* of a nilpotent Lie algebra  $\mathfrak{n}$  is the  $r$ -tuple  $(n_1, \dots, n_r)$ , where  $n_i = \dim C^{i-1}(\mathfrak{n})/C^i(\mathfrak{n})$  and  $C^i(\mathfrak{n})$  is the central descending series. By using that any Anosov Lie algebra admits an Anosov automorphism  $A$  which is semisimple and some elementary properties of lattices, one sees that only a few types are allowed in each dimension 7 and 8. We then study these types case by case in Section 4 and exploit that the eigenvalues of  $A$  are algebraic integers (even units). For each of the types we get only one or two real Lie algebras (sometimes no one at all) which are candidates to be Anosov, and some of them are excluded by using a criterion given in terms of a homogeneous polynomial (called the *Pfaffian form*) associated to each 2-step nilpotent Lie algebra.

We previously study the set of all rational forms up to isomorphism for each of the real Lie algebras obtained in the classification over  $\mathbb{R}$ . This is a subject on its own, and it is treated in Section 2, a part which is completely independent of the rest of the paper. The results obtained there (see Table 2) allows us to classify Anosov Lie algebras over  $\mathbb{Q}$  in Section 5, and here we also use a criterion on the Pfaffian form to discard some of them, which has in this case integer coefficients and hence some topics from number theory as the Pell equation and square free numbers appear. Such criterions and most of the known tools to deal with Anosov automorphisms are given in Section 3 (see also [4] for an approach via representation theory and arithmetic groups), as well as a generalization of the construction in [18] suggested by F. Grunewald, proving that  $\mathfrak{n} \oplus \dots \oplus \mathfrak{n}$  ( $s$  times,  $s \geq 2$ ) is Anosov for any graded nilpotent Lie algebra over  $\mathbb{R}$  having a rational form.

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## 2. RATIONAL FORMS OF NILPOTENT LIE ALGEBRAS

Since the classification of all nilmanifolds admitting an Anosov diffeomorphism reduces to the determination of a special class of nilpotent Lie algebras over  $\mathbb{Q}$ , we now start the study of rational forms of real nilpotent Lie algebras. Let  $\mathfrak{n}$  be a nilpotent Lie algebra over  $\mathbb{R}$  of dimension  $n$ .

**Definition 2.1.** A *rational form* of  $\mathfrak{n}$  is an  $n$ -dimensional rational subspace  $\mathfrak{n}^{\mathbb{Q}}$  of  $\mathfrak{n}$  such that

$$[X, Y] \in \mathfrak{n}^{\mathbb{Q}}, \quad \forall X, Y \in \mathfrak{n}^{\mathbb{Q}}.$$

Two rational forms  $\mathfrak{n}_1^{\mathbb{Q}}, \mathfrak{n}_2^{\mathbb{Q}}$  of  $\mathfrak{n}$  are said to be *isomorphic* if there exists  $A \in \text{Aut}(\mathfrak{n})$  such that  $A\mathfrak{n}_1^{\mathbb{Q}} = \mathfrak{n}_2^{\mathbb{Q}}$ , or equivalently, if they are isomorphic as Lie algebras over  $\mathbb{Q}$  (recall that  $\mathfrak{n}^{\mathbb{Q}} \otimes \mathbb{R} = \mathfrak{n}$ ). Analogously, by considering  $\mathbb{R}$  and  $\mathbb{C}$  (resp.  $\mathbb{Q}$  and  $\mathbb{C}$ ) instead of  $\mathbb{Q}$  and  $\mathbb{R}$  one defines a *real form* (resp. a *rational form*) of a complex Lie algebra.

Not every real nilpotent Lie algebra admits a rational form. By a result due to Malcev, the existence of a rational form of  $\mathfrak{n}$  is equivalent to the corresponding Lie group  $N$  admits a *lattice*, i.e. a cocompact discrete subgroup (see [25]). Another difference with the semisimple case is that sometimes  $\mathfrak{n}$  has only one rational form up to isomorphism. The problem of finding all isomorphism classes of rational forms for a given real nilpotent Lie algebra is a very difficult one, even in the low dimensional or two-step cases. Very little is known about this challenge problem in the literature (see [8, Section 5]). When  $\mathfrak{n}$  is two-step nilpotent and has 2-dimensional center, F. Grunewald and D. Segal [13, 14] gave an answer in terms of isomorphism classes of binary forms, which will be explained below. In [30] it is proved that  $\mathfrak{h}_{2k+1} \oplus \mathbb{R}^m$  has only one rational form up to isomorphism for all  $k, m$ , and that certain real Lie algebras of the form  $\mathfrak{g} \oplus \mathfrak{g}$  have infinitely many ones. In this section, we will find all the rational forms up to isomorphism of four real nilpotent Lie algebras of dimension 8 (see Table 2). This information will be useful in the classification of 8-dimensional Anosov Lie algebras.

Let  $\mathfrak{n}$  be a Lie algebra over the field  $K$ , which is assumed from now on to be of characteristic zero. We are mainly interested in the cases  $K = \mathbb{C}, \mathbb{R}, \mathbb{Q}$ . Fix a positive definite symmetric  $K$ -bilinear form  $\langle \cdot, \cdot \rangle$  on  $\mathfrak{n}$  (i.e. an inner product). For each  $Z \in \mathfrak{n}$  consider the  $K$ -linear transformation  $J_Z : \mathfrak{n} \rightarrow \mathfrak{n}$  defined by

$$(1) \quad \langle J_Z X, Y \rangle = \langle [X, Y], Z \rangle, \quad \forall X, Y \in \mathfrak{n}.$$

Recall that  $J_Z$  is skew symmetric with respect to  $\langle \cdot, \cdot \rangle$  and the map  $J : \mathfrak{n} \rightarrow \mathfrak{so}(n, K)$  is  $K$ -linear, where  $n$  is the dimension of  $\mathfrak{n}$ . Equivalently, we may define these maps by fixing a basis  $\beta = \{X_1, \dots, X_n\}$  of  $\mathfrak{n}$  rather than an inner product in the following way:  $J_Z$  is the  $K$ -linear transformation whose matrix in terms of  $\beta$  is

$$\left( \sum_{k=1}^n c_{ij}^k x_k \right),$$

where  $[X_i, X_j] = \sum_{k=1}^n c_{ij}^k X_k$  and  $Z = \sum_{k=1}^n x_k X_k$ . This coincides with the first definition if one sets  $\langle X_i, X_j \rangle = \delta_{ij}$ .

If  $\mathfrak{n}$  and  $\mathfrak{n}'$  are two Lie algebras over  $K$  and  $\{J_Z\}, \{J'_Z\}$  are the corresponding maps relative to the inner products  $\langle \cdot, \cdot \rangle$  and  $\langle \cdot, \cdot \rangle'$  respectively, then it is not hard to see that a linear map  $A : \mathfrak{n} \rightarrow \mathfrak{n}'$  is a Lie algebra isomorphism if and only if

$$(2) \quad A^t J'_Z A = J_{A^t Z}, \quad \forall Z \in \mathfrak{n},$$

where  $A^t : \mathfrak{n}' \rightarrow \mathfrak{n}$  is given by  $\langle A^t X, Y \rangle = \langle X, AY \rangle$  for all  $X \in \mathfrak{n}', Y \in \mathfrak{n}$ .

**Definition 2.2.** Consider the central descendent series defined by  $C^0(\mathfrak{n}) = \mathfrak{n}$ ,  $C^i(\mathfrak{n}) = [\mathfrak{n}, C^{i-1}(\mathfrak{n})]$ . When  $C^r(\mathfrak{n}) = 0$  and  $C^{r-1}(\mathfrak{n}) \neq 0$ ,  $\mathfrak{n}$  is said to be  $r$ -step nilpotent, and we denote by  $(n_1, \dots, n_r)$  the *type* of  $\mathfrak{n}$ , where

$$n_i = \dim C^{i-1}(\mathfrak{n}) / C^i(\mathfrak{n}).$$

We also take a decomposition  $\mathfrak{n} = \mathfrak{n}_1 \oplus \dots \oplus \mathfrak{n}_r$ , a direct sum of vector spaces, such that  $C^i(\mathfrak{n}) = \mathfrak{n}_{i+1} \oplus \dots \oplus \mathfrak{n}_r$  for all  $i$ .

Assume now that  $\mathfrak{n}$  is 2-step nilpotent, or equivalently of type  $(n_1, n_2)$ . We will always have fixed orthonormal basis  $\{X_i\}$  and  $\{Z_j\}$  of  $\mathfrak{n}_1$  and  $\mathfrak{n}_2$ , respectively. Consider any direct sum decomposition of the form  $\mathfrak{n} = V \oplus [\mathfrak{n}, \mathfrak{n}]$ , that is,  $\mathfrak{n}_1 = V$ . If the inner product satisfies  $\langle V, [\mathfrak{n}, \mathfrak{n}] \rangle = 0$  then  $V$  is  $J_Z$ -invariant for any  $Z$  and  $J_Z = 0$  if and only if  $Z \in V$ . We define  $f : [\mathfrak{n}, \mathfrak{n}] \mapsto K$  by

$$f(Z) = \text{Pf}(J_Z|_V), \quad Z \in [\mathfrak{n}, \mathfrak{n}],$$

where  $\text{Pf} : \mathfrak{so}(V, K) \mapsto K$  is the *Pfaffian*, that is, the only polynomial function satisfying  $\text{Pf}(B)^2 = \det B$  for all  $B \in \mathfrak{so}(V, K)$  and  $\text{Pf}(J) = 1$  for

$$J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}.$$

Roughly speaking,  $f(Z) = (\det J_Z|_V)^{\frac{1}{2}}$ , and so we need  $\dim V$  to be even in order to get  $f \neq 0$ . For any  $A \in \mathfrak{gl}(V, K)$ ,  $B \in \mathfrak{so}(V, K)$  we have that  $\text{Pf}(ABA^t) = (\det A) \text{Pf}(B)$ .

**Definition 2.3.** We call  $f$  the *Pfaffian form* of  $\mathfrak{n}$ .

If  $\dim V = 2m$  and  $\dim [\mathfrak{n}, \mathfrak{n}] = k$  then  $f = f(x_1, \dots, x_k)$  is a homogeneous polynomial of degree  $m$  in  $k$  variables with coefficients in  $K$ , where  $Z = \sum_{i=1}^k x_i Z_i$  and  $\{Z_1, \dots, Z_k\}$  is a fixed basis of  $[\mathfrak{n}, \mathfrak{n}]$ .  $f$  is also called a form of degree  $m$ , when  $k = 2$  or  $3$  one uses the words binary or ternary and for  $m = 2, 3$  and  $4$ , quadratic, cubic and cuartic, respectively.

Let  $P_{k,m}(K)$  denote the set of all homogeneous polynomials of degree  $m$  in  $k$  variables with coefficients in  $K$ . The group  $GL_k(K)$  acts naturally on  $P_{k,m}(K)$  by

$$(A.f)(x_1, \dots, x_k) = f(A^{-1}(x_1, \dots, x_k)),$$

that is, by linear substitution of variables, and thus the action determines the usual equivalence relation between forms, denoted by  $f \simeq g$ . In the present paper, we need to consider the following wider equivalence relation.

**Definition 2.4.** For  $f, g \in P_{k,m}(K)$ , we say that  $f$  is *projectively equivalent* to  $g$ , and denote it by  $f \simeq_K g$ , if there exists  $A \in GL_k(K)$  and  $c \in K^*$  such that

$$f(x_1, \dots, x_k) = cg(A(x_1, \dots, x_k)).$$

In other words, we are interested in projective equivalence classes of forms.

**Proposition 2.5.** *Let  $\mathfrak{n}, \mathfrak{n}'$  be two-step nilpotent Lie algebras over the field  $K$ . If  $\mathfrak{n}$  and  $\mathfrak{n}'$  are isomorphic then  $f \simeq_K f'$ , where  $f$  and  $f'$  are the Pfaffian forms of  $\mathfrak{n}$  and  $\mathfrak{n}'$ , respectively.*

*Proof.* Since  $\mathfrak{n}$  and  $\mathfrak{n}'$  are isomorphic we can assume that  $\mathfrak{n} = \mathfrak{n}'$  and  $[\mathfrak{n}, \mathfrak{n}] = [\mathfrak{n}', \mathfrak{n}']$  as vector spaces, and then the decomposition  $\mathfrak{n} = V \oplus [\mathfrak{n}, \mathfrak{n}]$  is valid for both Lie brackets  $[\cdot, \cdot]$  and  $[\cdot, \cdot]'$ . Any isomorphism satisfies  $A[\mathfrak{n}, \mathfrak{n}] = [\mathfrak{n}', \mathfrak{n}']'$ , and it is easy

to see that there is always an isomorphism  $A$  between them satisfying  $AV = V$ . It follows from (2) that

$$A^t J'_Z A = J_{A^t Z}, \quad \forall Z \in [\mathfrak{n}, \mathfrak{n}],$$

and since the subspaces  $V$  and  $[\mathfrak{n}, \mathfrak{n}]$  are preserved by  $A$  and  $A^t$  we have that

$$f'(Z) = cf(A_2^t Z),$$

where  $A_2 = A|_{[\mathfrak{n}, \mathfrak{n}]}$  and  $c^{-1} = \det A|_V$ . This shows that  $f \simeq_K f'$ .  $\square$

The above proposition says that the (projective) equivalence class of the form  $f(x_1, \dots, x_k)$  is an isomorphism invariant of the Lie algebra  $\mathfrak{n}$ . This invariant was introduced by Scheuneman in [26].

What is known about the classification of forms? Unfortunately, much less than one could naively expect. The case  $K = \mathbb{C}$  is as usual the most developed one, and there the understanding of the ring of invariant polynomials  $\mathbb{C}[P_{k,m}]^{SL_k(\mathbb{C})}$  is crucial. A set of generators and their relations for such a ring is known only for small values of  $k$  and  $m$ , for instance for  $k = 2$  and  $m \leq 8$ , or  $k = 3$  and  $m \leq 3$ . The following well known result will help us to distinguish between projective equivalence classes of forms, and in view of Proposition 2.5, to recognize non-isomorphic two-step nilpotent Lie algebras.

**Proposition 2.6.** *If  $f, g \in P_{k,m}(K)$  satisfy*

$$f(x_1, \dots, x_k) = cg(A(x_1, \dots, x_k))$$

*for some  $A \in GL_k(K)$  and  $c \in K^*$ , then*

$$Hf(x_1, \dots, x_k) = c^k (\det A)^2 Hg(A(x_1, \dots, x_k)),$$

*where the Hessian  $Hf$  of the form  $f$  is defined by*

$$Hf(x_1, \dots, x_k) = \det \left[ \frac{\partial^2 f}{\partial x_i \partial x_j} \right].$$

Let  $\mathfrak{n}^{\mathbb{Q}}$  be a rational nilpotent Lie algebra of type  $(4, 2)$ . If  $\mathfrak{n}^{\mathbb{Q}} = \mathfrak{n}_1 \oplus \mathfrak{n}_2$  is the decomposition such that  $\dim \mathfrak{n}_1 = 4$ ,  $\dim \mathfrak{n}_2 = 2$  and  $[\mathfrak{n}^{\mathbb{Q}}, \mathfrak{n}^{\mathbb{Q}}] = \mathfrak{n}_2$ , then we consider the Pfaffian form  $f$  of  $\mathfrak{n}$ . Thus  $f$  is a binary quadratic form, say  $f(x, y) = ax^2 + bxy + cy^2$ , with  $a, b, c \in \mathbb{Q}$ . The strong result proved in [13] is that the converse of Proposition 2.5 is valid in this case, that is, there is a one-to-one correspondence between isomorphism classes of non-degenerate (i.e. with center equal to  $\mathfrak{n}_2$ ) rational Lie algebras of type  $(4, 2)$  and projective equivalence classes of binary quadratic forms with coefficients in  $\mathbb{Q}$ . It is easy to see that such classes can be parametrized by

$$\{f_k(x, y) = x^2 - ky^2 : k \text{ is a square free integer number}\}.$$

Recall that an integer number is said to be *square free* if  $p^2 \nmid k$  for any prime  $p$ , and the set of all square free numbers parametrizes the equivalence classes of the relation in  $\mathbb{Q}$  defined by  $r \equiv s$  if and only if  $r = q^2 s$  for some  $q \in \mathbb{Q}^*$ . We are considering  $k = 0$  a square free number too. If  $f_k \simeq_K f_{k'}$  then it follows from Proposition 2.6 that  $-4k = -4q^2 k'$  for some  $q \in \mathbb{Q}^*$ , which implies that  $k = k'$  in the case  $k$  and  $k'$  are square free.

It is not hard to prove that the Pfaffian form of the Lie algebra  $\mathfrak{n}_k^{\mathbb{Q}} = \mathfrak{n}_1 \oplus \mathfrak{n}_2$  defined by

$$(3) \quad [X_1, X_3] = Z_1, \quad [X_1, X_4] = Z_2, \quad [X_2, X_3] = kZ_2, \quad [X_2, X_4] = Z_1$$

is  $f_k$ . For  $K = \mathbb{R}$ , these Lie algebras can be distinguished only by the sign of the discriminant of  $f_k$ , which says that there are only three real Lie algebras of type  $(4, 2)$ , namely, those of the form  $\mathfrak{n}_k^{\mathbb{Q}} \otimes \mathbb{R}$  with  $k > 0$ ,  $k = 0$  and  $k < 0$ . We have that  $\mathfrak{n}_1^{\mathbb{Q}} \otimes \mathbb{R} \simeq \mathfrak{h}_3 \oplus \mathfrak{h}_3$ , where  $\mathfrak{h}_3$  denotes the 3-dimensional Heisenberg Lie algebra and  $\mathfrak{n}_{-1}^{\mathbb{Q}} \otimes \mathbb{R}$  is an H-type Lie algebra. Analogously, there are only two complexifications  $\mathfrak{n}_k^{\mathbb{Q}} \otimes \mathbb{C}$ , those with  $k \neq 0$  and  $k = 0$ .

**Proposition 2.7.** *The set of isomorphism classes of rational forms of the Lie algebra  $\mathfrak{h}_3 \oplus \mathfrak{h}_3$  is parametrized by*

$$\{\mathfrak{n}_k^{\mathbb{Q}} : k \text{ is a square free natural number}\}.$$

*Proof.* The Lie bracket of  $\mathfrak{h}_3 \oplus \mathfrak{h}_3$  is

$$[X_1, X_2] = Z_1, \quad [X_3, X_4] = Z_2,$$

and one can easily check that the rational subspace generated by the set

$$\left\{ X_1 + X_3, \sqrt{k}(X_1 - X_3), \sqrt{k}(X_2 + X_4), X_2 - X_4, \sqrt{k}(Z_1 + Z_2), Z_1 - Z_2 \right\},$$

is a rational subalgebra of  $\mathfrak{h}_3 \oplus \mathfrak{h}_3$  isomorphic to  $\mathfrak{n}_k^{\mathbb{Q}}$ .  $\square$

We now describe the results in [14] for the general case (see also [11]). Consider  $\mathfrak{n} = \mathfrak{n}_1 \oplus \mathfrak{n}_2$  a vector space over  $K$  such that  $\mathfrak{n}_1$  and  $\mathfrak{n}_2$  are subspaces of dimension  $n$  and 2 respectively. Every 2-step nilpotent Lie algebra of dimension  $n + 2$  and 2-dimensional center can be represented by a bilinear form  $\mu : \mathfrak{n}_1 \times \mathfrak{n}_1 \rightarrow \mathfrak{n}_2$  which is non-degenerate in the following way: for any nonzero  $X \in \mathfrak{n}_1$  there is a  $Y \in \mathfrak{n}_1$  such that  $\mu(X, Y) \neq 0$ . If we fix basis  $\{X_1, \dots, X_n\}$  and  $\{Z_1, Z_2\}$  of  $\mathfrak{n}_1$  and  $\mathfrak{n}_2$  respectively, then each  $\mu$  has an associated Pfaffian binary form  $f_\mu$  defined by

$$f_\mu(x, y) = \text{Pf}(J_{xZ_1+yZ_2}^\mu)$$

(see Definition 2.3). A *central decomposition* of  $\mu$  is given by a decomposition of  $\mathfrak{n}_1$  in a direct sum of subspaces  $\mathfrak{n}_1 = V_1 \oplus \dots \oplus V_r$  such that  $\mu(V_i, V_j) = 0$  for all  $i \neq j$ . We say that  $\mu$  is *indecomposable* when the only possible central decomposition has  $r = 1$ . Every  $\mu$  has a central decomposition into indecomposable constituents and such a decomposition is unique up to an automorphism of  $\mu$ ; in particular, the constituents  $V_i \oplus \mathfrak{n}_2$  are unique up to isomorphism.

There is only one indecomposable  $\mu$  for  $n$  odd and it can be defined by

$$J_{xZ_1+yZ_2}^\mu = \left[ \begin{array}{ccc|cccc} & & & -x & -y & & & 0 \\ & & & 0 & -x & -y & & \\ & & & & & \ddots & \ddots & \\ & & & 0 & & & -x & -y \\ x & 0 & 0 & & & & & \\ y & x & & & & & & \\ 0 & y & \ddots & & & & & \\ & & \ddots & x & & & & \\ 0 & & & y & & & & \end{array} \right].$$

Recall that  $f_\mu = 0$  in this case. When  $n$  is even the situation is much more abundant: two indecomposables  $\mu$  and  $\lambda$  are isomorphic if and only if  $f_\mu \simeq_K f_\lambda$ . If  $n = 2m$  and  $f_\mu(x, y) = x^m - a_1 x^{m-1} y - \dots - a_m y^m$ , then

$$J_{xZ_1+yZ_2}^\mu = \begin{bmatrix} 0 & -B^t \\ B & 0 \end{bmatrix},$$

where

$$B = \begin{bmatrix} x & y & 0 & \cdots & 0 \\ 0 & x & y & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & x & y \\ a_m y & a_{m-1} y & \cdots & a_2 y & a_1 y + x \end{bmatrix}.$$

We note that here  $f_\mu$  is always nonzero, and in order to get  $\mu$  indecomposable one needs the form  $f_\mu$  to be primitive (i.e. a power of an irreducible one). For decomposable  $\mu$  and  $\lambda$  with respective central decompositions  $\mathfrak{n}_1 = V_1 \oplus \dots \oplus V_r$  and  $\mathfrak{n}_1 = W_1 \oplus \dots \oplus W_s$  into indecomposables constituents, we have that  $\mu$  is isomorphic to  $\lambda$  if and only if  $r = s$  and after a suitable reordering one has that

- (i) for some  $t \leq r$ ,  $\dim V_i = \dim W_i$  for all  $i = 1, \dots, t$  and they are all even numbers;
- (ii) if  $\mu_i = \mu|_{V_i \times V_i}$ ,  $\lambda_i = \lambda|_{W_i \times W_i}$  then there exist  $A \in GL_2(K)$  and  $c_1, \dots, c_t \in K^*$  such that

$$f_{\mu_i}(x, y) = c_i f_{\lambda_i}(A(x, y)) \quad \forall i = 1, \dots, t;$$

- (iii)  $\dim V_i = \dim W_i$  is odd for all  $i = t+1, \dots, r$ .

Concerning our search for all rational forms up to isomorphism of a given real nilpotent Lie algebra, these results say that the picture in the 2-step nilpotent with 2-dimensional center case is as follows. Let  $(\mathfrak{n}^\mathbb{Q} = \mathfrak{n}_1 \oplus \mathfrak{n}_2, \mu)$  be one of such Lie algebras over  $\mathbb{Q}$ , and consider the corresponding Pfaffian form  $f_\mu \in P_{2,m}(\mathbb{Q})$ . The isomorphism classes of rational forms of  $\mathfrak{n}^\mathbb{Q} \otimes \mathbb{R}$  are then parametrized by

$$((\mathbb{R}^* \times GL_2(\mathbb{R})).f_\mu \cap P_{2,m}(\mathbb{Q})) / (\mathbb{Q}^* \times GL_2(\mathbb{Q})).$$

In other words, the rational points of the orbit  $(\mathbb{R}^* \times GL_2(\mathbb{R})).f_\mu$  ( $f_\mu$  viewed as an element of  $P_{2,m}(\mathbb{R})$ ) is a  $(\mathbb{Q}^* \times GL_2(\mathbb{Q}))$ -invariant set and we have to consider the orbit space of this action. Such a description shows the high difficulty of the problem. Recall that we have to consider the action of  $\mathbb{R}^* \times GL_2(\mathbb{R})$  instead of just that of  $GL_2(\mathbb{R})$  only when  $m$  is even.

In what follows, we will study rational forms of four 8-dimensional nilpotent Lie algebras. We refer to Table 2 for a summary of the results obtained.

Let  $\mathfrak{g}$  be the 8-dimensional 2-step nilpotent Lie algebra defined by

$$(4) \quad [X_1, X_2] = Z_1, \quad [X_1, X_3] = Z_2, \quad [X_4, X_5] = Z_1, \quad [X_4, X_6] = Z_2.$$

It is easy to see that its Pfaffian form  $f$  is zero. Let  $\mathfrak{g}^\mathbb{Q}$  be a rational form of  $\mathfrak{g}$ , for which we can assume that  $\mathfrak{g}^\mathbb{Q} = \langle X_1, \dots, X_6 \rangle_\mathbb{Q} \oplus \langle Z_1, Z_2 \rangle_\mathbb{Q}$ . Since the Pfaffian form  $g$  of  $\mathfrak{g}^\mathbb{Q}$  satisfies  $g \simeq_\mathbb{R} f = 0$  we obtain that  $g = 0$ . By using the results described above one deduces that  $\mathfrak{g}^\mathbb{Q}$  can not be indecomposable, and so  $\langle X_1, \dots, X_6 \rangle_\mathbb{Q} = V_1 \oplus \dots \oplus V_r$  with  $[V_i, V_j] = 0$  for all  $i \neq j$ . Now,  $\langle X_1, \dots, X_6 \rangle_\mathbb{R} = V_1 \otimes \mathbb{R} \oplus \dots \oplus V_r \otimes \mathbb{R}$  is also a central decomposition for  $\mathfrak{g}$ , proving that  $r = 2$  and  $\dim V_1 = \dim V_2 = 3$  by the

<i>Notation</i>	<i>Type</i>	<i>Lie brackets</i>
$\mathfrak{h}_{2k+1}$	$(2k, 1)$	$[X_1, X_2] = Z_1, \dots, [X_{2k-1}, X_{2k}] = Z_1$
$\mathfrak{f}_3$	$(3, 3)$	$[X_1, X_2] = Z_1, [X_1, X_3] = Z_2, [X_2, X_3] = Z_3$
$\mathfrak{g}$	$(6, 2)$	$[X_1, X_2] = Z_1, [X_1, X_3] = Z_2, [X_4, X_5] = Z_1, [X_4, X_6] = Z_2$
$\mathfrak{h}$	$(4, 4)$	$[X_1, X_3] = Z_1, [X_1, X_4] = Z_2, [X_2, X_3] = Z_3, [X_2, X_4] = Z_4$
$\mathfrak{l}_4$	$(2, 1, 1)$	$[X_1, X_2] = X_3, [X_1, X_3] = X_4$

TABLE 1. Notation for some real nilpotent Lie algebras.

uniqueness of such a decomposition. By applying again the results described above, now to the odd situation, we get the following

**Proposition 2.8.** *The Lie algebra  $\mathfrak{g}$  of type  $(6, 2)$  given in (4) has only one rational form up to isomorphism, denoted by  $\mathfrak{g}^{\mathbb{Q}}$ .*

**Remark 2.9.** Clearly, the same proof is valid if one need to find all real forms of the complex Lie algebra  $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes \mathbb{C}$ . Thus  $\mathfrak{g}$  is the only real form of  $\mathfrak{g}_{\mathbb{C}}$  up to isomorphism.

As another application of the correspondence with binary forms given above, we now study rational forms of the real Lie algebra  $\mathfrak{h}_3 \oplus \mathfrak{h}_5$  of type  $(6, 2)$ . It has central decomposition  $\mathfrak{n}_1 = V_1 \oplus V_2 \oplus V_3$  with  $\dim V_i = 2$  for all  $i$  as a real Lie algebra and its Pfaffian form is  $f(x, y) = xy^2$ . Let  $\mu : \mathfrak{n}_1 \times \mathfrak{n}_1 \mapsto \mathfrak{n}_2$  be a rational form of  $\mathfrak{h}_3 \oplus \mathfrak{h}_5$  with Pfaffian form  $f_{\mu}$ . If  $\mu$  is decomposable then  $\mathfrak{n}_1 = W_1 \oplus W_2$ ,  $\dim W_1 = 2$ ,  $\dim W_2 = 4$ ; or  $\mathfrak{n}_1 = W_1 \oplus W_2 \oplus W_3$ ,  $\dim W_i = 2$  for all  $i$ . In any case,  $f_{\mu_i} \simeq_{\mathbb{Q}} x, y$  or  $y^2$  proving that  $\mu$  must be isomorphic to the canonical rational form

$$\mu_0(X_1, X_2) = Z_1, \quad \mu_0(X_3, X_4) = Z_2, \quad \mu_0(X_5, X_6) = Z_2,$$

for which  $f_{\mu_0} = f$ . We then assume that  $\mu$  is indecomposable. We shall prove that there is only one  $GL_2(\mathbb{Q})$ -orbit of rational points in  $GL_2(\mathbb{R}).f$ , and so  $\mu$  will have to be isomorphic to  $\mu_0$ . There exists  $A \in GL_2(\mathbb{R})$  such that  $f_{\mu} = A^{-1}.f$ , that is,

$$f_{\mu}(x, y) = ac^2x^3 + c(2ad + bc)x^2y + d(ad + 2bc)xy^2 + bd^2y^3, \quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Since  $\mu$  is rational we have that

$$q := ac^2, \quad r := c(2ad + bc), \quad s := d(ad + 2bc), \quad t := bd^2$$

are all in  $\mathbb{Q}$ . If  $c = 0$  then  $q = r = 0$  and  $s = ad^2$ ,  $t = bd^2$ , which implies that  $s \neq 0$  and hence

$$f_{\mu} = B^{-1}.f, \quad \text{for } B = \begin{bmatrix} s & t \\ 0 & 1 \end{bmatrix} \in GL_2(\mathbb{Q}).$$

If  $c \neq 0$  then one can check by a straightforward computation that

$$\frac{d}{c} = \frac{9qst + rs^2 - 6r^2t}{6qs^2 - r^2s - 9qrt} \in \mathbb{Q}.$$

There must be a simpler formula for  $\frac{d}{c}$  in terms of  $q, r, s, t$ , but unfortunately we were not able to find it. By putting  $u := \frac{d}{c}$  we have that

$$f_\mu = B^{-1} \cdot f, \quad \text{for } B = \begin{bmatrix} q & t/u^2 \\ 1 & u \end{bmatrix} \in GL_2(\mathbb{Q}).$$

Recall that  $\det B = qu - \frac{t}{u^2} = c(ad - bc) = c \det A \neq 0$ . We then obtain that in any case  $f_\mu \simeq_{\mathbb{Q}} f$  and so  $\mu$  is isomorphic to  $\mu_0$ .

**Proposition 2.10.** *Up to isomorphism, the real Lie algebra  $\mathfrak{h}_3 \oplus \mathfrak{h}_5$  of type (6, 2) has only one rational form, which will be denoted by  $(\mathfrak{h}_3 \oplus \mathfrak{h}_5)^{\mathbb{Q}}$ .*

**Remark 2.11.** It is easy to check that the above proof is also valid if we replace  $\mathbb{Q}$  and  $\mathbb{R}$  by  $\mathbb{R}$  and  $\mathbb{C}$ , obtaining in this way that the only real form of  $(\mathfrak{h}_3 \oplus \mathfrak{h}_5)_{\mathbb{C}}$  is  $\mathfrak{h}_3 \oplus \mathfrak{h}_5$ .

We now describe a duality for 2-step nilpotent Lie algebras over any field of characteristic zero introduced by J. Scheuneman [26] (see also [11]), which assigns to each Lie algebra of type  $(n, k)$  another one of type  $(n, \frac{n(n-1)}{2} - k)$ . The dual of a Lie algebra  $\mathfrak{n} = \mathfrak{n}_1 \oplus \mathfrak{n}_2$  of type  $(n, k)$  can be defined as follows: consider the maps  $\{J_Z : Z \in \mathfrak{n}_2\} \subset \mathfrak{so}(n)$  corresponding to a fixed inner product  $\langle \cdot, \cdot \rangle$  on  $\mathfrak{n}$  (see (1)). Let  $\tilde{\mathfrak{n}}_2 \subset \mathfrak{so}(n)$  be the orthogonal complement of the  $k$ -dimensional subspace  $\{J_Z : Z \in \mathfrak{n}_2\}$  in  $\mathfrak{so}(n)$  relative to the inner product  $(A, B) = -\text{tr } AB$ . Now, we define the 2-step nilpotent Lie algebra  $\tilde{\mathfrak{n}} = \mathfrak{n}_1 \oplus \tilde{\mathfrak{n}}_2$  whose Lie bracket is determined by

$$([X, Y], Z) = \langle Z(X), Y \rangle, \quad Z \in \tilde{\mathfrak{n}}_2.$$

In other words, the maps  $\tilde{J}_Z$ 's for this Lie algebra are the  $Z$ 's themselves. Recall that  $\dim \tilde{\mathfrak{n}}_2 = \frac{n(n-1)}{2} - k$ , and so the dual  $\tilde{\mathfrak{n}}$  of  $\mathfrak{n}$  is of type  $(n, \frac{n(n-1)}{2} - k)$ . It is proved in [26] that  $\mathfrak{n}_1$  is isomorphic to  $\mathfrak{n}_2$  if and only if  $\tilde{\mathfrak{n}}_1$  is isomorphic to  $\tilde{\mathfrak{n}}_2$ , so that any classification of type  $(n, k)$  simultaneously classifies type  $(n, \frac{n(n-1)}{2} - k)$ .

**Example 2.12.** Let  $\mathfrak{h}$  be the Lie algebra of type (4, 4) which is dual to  $\mathfrak{h}_3 \oplus \mathfrak{h}_3$  (of type (4, 2)). The Lie bracket of  $\mathfrak{h}_3 \oplus \mathfrak{h}_3$  is

$$[X_1, X_2] = Z_1, \quad [X_3, X_4] = Z_2,$$

and hence

$$J_{Z_1} = \begin{bmatrix} 0 & -1 & & \\ 1 & 0 & & \\ & & 0 & 0 \\ & & 0 & 0 \end{bmatrix}, \quad J_{Z_2} = \begin{bmatrix} 0 & 0 & & \\ 0 & 0 & & \\ & & 0 & -1 \\ & & 1 & 0 \end{bmatrix}.$$

The orthogonal complement  $\tilde{\mathfrak{n}}_2$  of  $\{J_Z : Z \in \mathfrak{n}_2\}$  is then linearly generated by

$$\begin{bmatrix} 1 & 0 & & \\ 0 & 0 & & \\ & & -1 & 0 \\ & & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & & \\ 0 & 0 & & \\ & & 0 & -1 \\ & & 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & & \\ 0 & 1 & & \\ & & -1 & 0 \\ & & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & & \\ 0 & 0 & & \\ & & 0 & -1 \\ & & 1 & 0 \end{bmatrix},$$

which determines the Lie bracket for  $\mathfrak{h}$  given by

$$(5) \quad [X_1, X_3] = Z_1, \quad [X_1, X_4] = Z_2, \quad [X_2, X_3] = Z_3, \quad [X_2, X_4] = Z_4.$$

Scheuneman duality also allows us to find all the rational forms of  $\mathfrak{h}$ ; namely, the duals of the rational form of  $\mathfrak{h}_3 \oplus \mathfrak{h}_3$ , already computed in Proposition 2.7.

**Proposition 2.13.** *For any  $k \in \mathbb{Z}$  let  $\mathfrak{h}_k^{\mathbb{Q}}$  be the rational Lie algebra of type (4, 4) defined by*

$$\begin{aligned} [X_1, X_2] &= Z_1, & [X_2, X_3] &= -Z_3, \\ [X_1, X_3] &= Z_2, & [X_2, X_4] &= -Z_2, \\ [X_1, X_4] &= kZ_3, & [X_3, X_4] &= Z_4. \end{aligned}$$

Then the set of isomorphism classes of rational forms of the Lie algebra  $\mathfrak{h}$  defined in (5) is parametrized by

$$\{\mathfrak{h}_k^{\mathbb{Q}} : k \text{ is a square free natural number}\}.$$

*Proof.* For the rational form  $\mathfrak{n}_k^{\mathbb{Q}}$  of  $\mathfrak{h}_3 \oplus \mathfrak{h}_3$  (see (3)) we have that

$$J_{Z_1} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad J_{Z_2} = \begin{bmatrix} 0 & -1 \\ 0 & k \\ 1 & 0 \end{bmatrix}.$$

A basis of the orthogonal complement of  $\langle J_{Z_1}, J_{Z_2} \rangle_{\mathbb{Q}}$  is then given by

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & -1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & -k \\ 0 & 1 \\ k & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & -1 \\ 1 & 0 \end{bmatrix},$$

which determines the Lie bracket for  $\mathfrak{h}_k^{\mathbb{Q}}$ . To conclude the proof, one can easily check that the rational subspace generated by

$$\left\{ \sqrt{k}(X_1 - X_3), X_1 + X_3, X_2 + X_4, \sqrt{k}(X_2 - X_4), \right. \\ \left. 2\sqrt{k}Z_1, \sqrt{k}(Z_2 + Z_3), Z_3 - Z_2, -2\sqrt{k}Z_4 \right\},$$

is closed under the Lie bracket of  $\mathfrak{h}$  and isomorphic to  $\mathfrak{h}_k^{\mathbb{Q}}$ .  $\square$

An alternative proof of the non-isomorphism between the  $\mathfrak{h}_k^{\mathbb{Q}}$ 's without using Scheuneman duality may be given as follows: from the form of  $J_{Z_1}, \dots, J_{Z_4}$  for  $\mathfrak{h}_k^{\mathbb{Q}}$  in the above proof it follows that

$$J_{xZ_1+yZ_2+zZ_3+wZ_4} = \begin{bmatrix} 0 & -x & -y & -kz \\ x & 0 & z & y \\ y & -z & 0 & -w \\ kz & -y & w & 0 \end{bmatrix},$$

and so the Pfaffian form of  $\mathfrak{h}_k^{\mathbb{Q}}$  is given by  $f_k(x, y, z, w) = xw + y^2 - kz^2$ . Now, if  $\mathfrak{h}_k^{\mathbb{Q}}$  is isomorphic to  $\mathfrak{h}_{k'}^{\mathbb{Q}}$  then  $f_k \simeq_{\mathbb{Q}} f_{k'}$  (see Proposition 2.5), which implies that  $k = q^2 k'$  for some  $q \in \mathbb{Q}^*$  by applying Proposition 2.6 (recall that  $Hf_k = 4k$ ). Thus  $k = k'$  since they are square free.

We now study rational forms of  $\mathfrak{l}_4 \oplus \mathfrak{l}_4$ , where  $\mathfrak{l}_4$  is the 4-dimensional real Lie algebra with Lie bracket

$$[Y_1, Y_2] = Y_3, \quad [Y_1, Y_3] = Y_4.$$

Since  $\mathfrak{l}_4 \oplus \mathfrak{l}_4$  is 3-step nilpotent, Pfaffian forms and duality can not be used as tools to distinguish or classify rational forms, which makes of this case the hardest one. For each  $k \in \mathbb{Z}$ , consider the 8-dimensional rational nilpotent Lie algebra  $\mathfrak{l}_k^{\mathbb{Q}}$  with basis  $\{X_1, X_2, X_3, X_4, Z_1, Z_2, Z_3, Z_4\}$  and Lie bracket defined by

$$(6) \quad \begin{aligned} [X_1, X_3] &= Z_1, & [X_2, X_3] &= Z_2, \\ [X_1, X_4] &= Z_2, & [X_2, X_4] &= kZ_1, \\ [X_1, Z_1] &= Z_3, & [X_2, Z_2] &= kZ_3, \\ [X_1, Z_2] &= Z_4, & [X_2, Z_1] &= Z_4. \end{aligned}$$

**Theorem 2.14.** *Let  $\{X_1, X_2, X_3, X_4, Z_1, Z_2, Z_3, Z_4\}$  be a basis of the Lie algebra  $\mathfrak{l}_4 \oplus \mathfrak{l}_4$  of type  $(4, 2, 2)$  with structure coefficients*

$$\begin{aligned} [X_1, X_3] &= Z_1, & [X_2, X_4] &= Z_2, \\ [X_1, Z_1] &= Z_3, & [X_2, Z_2] &= Z_4. \end{aligned}$$

<i>Real Lie algebra</i>	<i>Type</i>	<i>Rational forms</i>	<i>Reference</i>
$\mathfrak{h}_3 \oplus \mathfrak{h}_3$	(4, 2)	$\mathfrak{n}_k^{\mathbb{Q}}, k \geq 1$	Prop. 2.7
$\mathfrak{f}_3$	(3, 3)	$\mathfrak{f}_3^{\mathbb{Q}}$	--
$\mathfrak{g}$	(6, 2)	$\mathfrak{g}^{\mathbb{Q}}$	Prop. 2.8
$\mathfrak{h}_3 \oplus \mathfrak{h}_5$	(6, 2)	$(\mathfrak{h}_3 \oplus \mathfrak{h}_5)^{\mathbb{Q}}$	Prop. 2.10
$\mathfrak{h}$	(4, 4)	$\mathfrak{h}_k^{\mathbb{Q}}, k \geq 1$	Prop. 2.13
$\mathfrak{l}_4 \oplus \mathfrak{l}_4$	(4, 2, 2)	$\mathfrak{l}_k^{\mathbb{Q}}, k \geq 1$	Prop. 2.14

TABLE 2. Set of rational forms up to isomorphism for some real nilpotent Lie algebras. In all cases  $k$  runs over all square-free natural numbers.

For each  $k \in \mathbb{N}$  the rational subspace generated by the set

$$\left\{ X_1 + X_2, \sqrt{k}(X_1 - X_2), X_3 + X_4, \sqrt{k}(X_3 - X_4), \right. \\ \left. Z_1 + Z_2, \sqrt{k}(Z_1 - Z_2), Z_3 + Z_4, \sqrt{k}(Z_3 - Z_4) \right\}$$

is a rational form of  $\mathfrak{l}_4 \oplus \mathfrak{l}_4$  isomorphic to the Lie algebra  $\mathfrak{l}_k^{\mathbb{Q}}$  defined in (6). Moreover, the set

$$\{\mathfrak{l}_k^{\mathbb{Q}} : k \text{ is a square-free natural number}\}$$

parametrizes all the rational forms of  $\mathfrak{l}_4 \oplus \mathfrak{l}_4$  up to isomorphism.

*Proof.* It is easy to see that the Lie brackets of the basis of the rational subspace coincides with the one of  $\mathfrak{l}_k^{\mathbb{Q}}$  by renaming the basis as  $\{X_1, \dots, Z_4\}$  with the same order. In particular, such a subspace is a rational form of  $\mathfrak{l}_4 \oplus \mathfrak{l}_4$ . If  $k' = q^2k$  then one can easily check that  $A : \mathfrak{l}_{k'}^{\mathbb{Q}} \mapsto \mathfrak{l}_k^{\mathbb{Q}}$  given by the diagonal matrix with entries  $(1, q, 1, q, 1, q, 1, q)$  is an isomorphism of Lie algebras.

Conversely, assume that  $A : \mathfrak{l}_k^{\mathbb{Q}} \mapsto \mathfrak{l}_{k'}^{\mathbb{Q}}$  is an isomorphism. We will show that  $k' = q^2k$  for some  $q \in \mathbb{Q}^*$ . Let  $\{J_Z'\}, \{J_Z\}$  be the maps defined at the beginning of this section corresponding to  $\mathfrak{l}_{k'}^{\mathbb{Q}}$  and  $\mathfrak{l}_k^{\mathbb{Q}}$ , respectively. If  $Z = xZ_1 + yZ_2 + zZ_3 + wZ_4$

we have that

$$J_Z = \begin{bmatrix} 0 & 0 & -x & -y & -z & -w & 0 & 0 \\ 0 & 0 & -y & -kx & -w & -kz & 0 & 0 \\ x & y & 0 & & \dots & & & 0 \\ y & kx & & & & & & \\ z & w & \vdots & & & & \vdots & \vdots \\ w & kz & & & & & & \\ 0 & 0 & & & & & & \\ 0 & 0 & 0 & & \dots & & & 0 \end{bmatrix},$$

and  $J'_Z$  is obtained just by replacing  $k$  with  $k'$ . It follows from (2) that  $A^t J'_Z A = J_{A^t Z}$  for all  $Z \in \langle Z_3, Z_4 \rangle_{\mathbb{Q}}$ , and since this subspace is  $A$ -invariant we get that the subspace

$$\bigcap_{Z \in \langle Z_3, Z_4 \rangle_{\mathbb{Q}}} \text{Ker } J_Z = \bigcap_{Z \in \langle Z_3, Z_4 \rangle_{\mathbb{Q}}} \text{Ker } J'_Z = \langle X_3, X_4, Z_3, Z_4 \rangle_{\mathbb{Q}}$$

is also  $A$ -invariant. Thus  $A$  has the form

$$(7) \quad A = \begin{bmatrix} A_1 & 0 & 0 & 0 \\ \star & A_2 & 0 & 0 \\ \star & 0 & A_3 & 0 \\ \star & \star & \star & A_4 \end{bmatrix}$$

(recall that  $C^1(\mathfrak{l}_k^{\mathbb{Q}}) = C^1(\mathfrak{l}_{k'}^{\mathbb{Q}}) = \langle Z_1, Z_2, Z_3, Z_4 \rangle_{\mathbb{Q}}$  and  $C^2(\mathfrak{l}_k^{\mathbb{Q}}) = C^2(\mathfrak{l}_{k'}^{\mathbb{Q}}) = \langle Z_3, Z_4 \rangle_{\mathbb{Q}}$  are always  $A$ -invariant), and now it is easy to prove that

$$A_3^t \begin{bmatrix} z & w \\ w & k'z \end{bmatrix} A_1 = \begin{bmatrix} az + bw & cz + dw \\ cz + dw & k'(az + bw) \end{bmatrix}, \text{ where } A_4^t = \begin{bmatrix} a & bw \\ c & d \end{bmatrix}.$$

We compute the determinant of both sides getting

$$qf'(z, w) = f(A_4^t(z, w)), \quad \forall (z, w) \in \mathbb{Q}^2,$$

where  $q = \det A_3 A_1 \in \mathbb{Q}^*$  and  $f(z, w) = kz^2 - w^2$ ,  $f'(z, w) = k'z^2 - w^2$ . By Proposition 2.6 we have that

$$4k' = q^{-2}(\det A_4)^2 4k,$$

and so  $k = k'$  as long as they are square free numbers, as we wanted to show.

To conclude the proof, it remains to show that these are all the rational forms up to isomorphism. Let  $\mathfrak{n}^{\mathbb{Q}}$  be a rational form of  $\mathfrak{l}_4 \oplus \mathfrak{l}_4$ . Since  $\mathfrak{n}^{\mathbb{Q}}/[\mathfrak{n}^{\mathbb{Q}}, [\mathfrak{n}^{\mathbb{Q}}, \mathfrak{n}^{\mathbb{Q}}]]$  is of type (4, 2), we can use the classification of rational Lie algebras of this type given in (3) to get linearly independent vectors  $X_1, \dots, Z_2$  such that

$$(8) \quad [X_1, X_3] = Z_1, \quad [X_1, X_4] = Z_2, \quad [X_2, X_3] = Z_2, \quad [X_2, X_4] = kZ_1,$$

where  $k$  is a square free integer number. Jacobi condition is equivalent to

$$(9) \quad \begin{aligned} [X_1, Z_2] &= [X_2, Z_1], & [X_3, Z_2] &= [X_4, Z_1], \\ k[X_1, Z_1] &= [X_2, Z_2], & k[X_3, Z_1] &= [X_4, Z_2]. \end{aligned}$$

We will consider the following two cases separately:

- (I)  $Z_3 := [X_1, Z_1]$  and  $Z_4 := [X_1, Z_2]$  are linearly independent,
- (II)  $[X_1, Z_1], [X_1, Z_2] \in \mathbb{Q}Z_3$  for some nonzero  $Z_3 \in \mathfrak{n}^{\mathbb{Q}}$ .

In both cases we will make use of the following isomorphism invariant for real 3-step nilpotent Lie algebras:

$$U(\mathfrak{n}) := \{X \in \mathfrak{n}/[\mathfrak{n}, [\mathfrak{n}, \mathfrak{n}]] : \dim \operatorname{Im}(\operatorname{ad} X) = 1\} \cup \{0\}.$$

Clearly, if  $A : \mathfrak{n} \rightarrow \mathfrak{n}'$  is an isomorphism then  $AU(\mathfrak{n}) = U(\mathfrak{n}')$ . Under the presentation of  $\mathfrak{l}_4 \oplus \mathfrak{l}_4$  given in the statement of the theorem, it is easy to see that

$$(10) \quad U(\mathfrak{l}_4 \oplus \mathfrak{l}_4) = \langle X_3, Z_1 \rangle_{\mathbb{R}} \cup \langle X_4, Z_3 \rangle_{\mathbb{R}}.$$

In case (I), it follows from (9) that we also have

$$[X_2, Z_1] = Z_4, \quad [X_2, Z_2] = kZ_3.$$

Therefore, in order to get that  $\mathfrak{n}^{\mathbb{Q}}$  is isomorphic to  $\mathfrak{l}_k^{\mathbb{Q}}$  (see (6)), it is enough to show that the vectors in  $\langle Z_3, Z_4 \rangle_{\mathbb{R}}$  given by

$$Z := k[X_3, Z_1] = [X_4, Z_2], \quad Z' := [X_3, Z_2] = [X_4, Z_1]$$

are both zero (see (9)). Let us compute the cone  $U(\mathfrak{n})$  for  $\mathfrak{n} = \mathfrak{n}^{\mathbb{Q}} \otimes \mathbb{R}$ . Recall that  $U(\mathfrak{n})$  has to be the union of two disjoint planes as  $\mathfrak{n} \simeq \mathfrak{l}_4 \oplus \mathfrak{l}_4$  (see (10)). If  $X = aX_1 + bX_2 + cX_3 + dX_4 + eZ_1 + fZ_2$  then

$$\begin{aligned} [X_1, X] &= cZ_1 + dZ_2 + eZ_3 + fZ_4, \\ [X_2, X] &= dkZ_1 + cZ_2 + fkZ_3 + eZ_4, \\ [X_3, X] &= -aZ_1 - bZ_2 + \frac{e}{k}Z + fZ', \\ [X_4, X] &= -bkZ_1 - aZ_2 + fZ + eZ', \\ [Z_1, X] &= -aZ_3 - bZ_4 - \frac{c}{k}Z - dZ', \\ [Z_2, X] &= -bkZ_3 - aZ_4 - dZ - cZ'. \end{aligned}$$

Assume that  $\operatorname{Im}(\operatorname{ad} X) = \mathbb{R}X_0$ ,  $X_0 \neq 0$ . If  $k \leq 0$  then it follows easily from  $[X_1, X] = \lambda[X_2, X]$  and  $[X_3, X] = \mu[X_4, X]$  for some  $\lambda, \mu \in \mathbb{R}$  that  $a = b = c = d = e = f = 0$ , which implies that  $U(\mathfrak{n}) = \{0\}$ , a contradiction.

**Remark 2.15.** Since  $k$  has to be positive one can also get by an easy adaptation of this proof that the only real form of  $(\mathfrak{l}_4 \oplus \mathfrak{l}_4)_{\mathbb{C}}$  is  $\mathfrak{l}_4 \oplus \mathfrak{l}_4$ .

We then have that  $k > 0$  and  $a = \pm\sqrt{k}b$ ,  $c = \pm\sqrt{k}d$ ,  $e = \pm\sqrt{k}f$ , where  $c$  and  $e$  have the same sign. This implies that

$$X = b(\pm\sqrt{k}X_1 + X_2) + d(\pm\sqrt{k}X_3 + X_4) + f(\pm\sqrt{k}Z_1 + Z_2)$$

and

$$\begin{aligned} [X_1, X] &= d(\pm\sqrt{k}Z_1 + Z_2) + f(\pm\sqrt{k}Z_3 + Z_4), \\ [X_2, X] &= \sqrt{k}[X_1, X], \\ [X_3, X] &= -b(\pm\sqrt{k}Z_1 + Z_2) + f(\pm\frac{1}{\sqrt{k}}Z + Z'), \\ [X_4, X] &= \sqrt{k}[X_3, X], \\ [Z_1, X] &= -b(\pm\sqrt{k}Z_3 + Z_4) - d(\pm\frac{1}{\sqrt{k}}Z + Z'), \\ [Z_2, X] &= \sqrt{k}[Z_1, X]. \end{aligned}$$

If  $b \neq 0$  then  $d \neq 0$  and  $a$  has the same sign as  $c$  and  $e$ , and since  $X_0$  has a nonzero component in  $\langle Z_1, Z_2 \rangle_{\mathbb{R}}$  we get  $[Z_1, X] = 0$ , that is,  $-\frac{b}{d}(\pm\sqrt{k}Z_3 + Z_4) = \pm\frac{1}{\sqrt{k}}Z + Z'$ . In any case we obtain a subset of  $U(\mathfrak{n})$  of the form

$$\{b(\pm\sqrt{k}X_1 + X_2) + d(\pm\sqrt{k}X_3 + X_4) + f(\pm\sqrt{k}Z_1 + Z_2) : b, d \neq 0\}$$

with the same sign in all the terms, which is a contradiction since  $U(\mathfrak{n})$  is the union of two planes. Thus  $b = 0$  and so

$$U(\mathfrak{n}) = \langle \sqrt{k}X_3 + X_4, \sqrt{k}Z_1 + Z_2 \rangle_{\mathbb{R}} \cup \langle -\sqrt{k}X_3 + X_4, -\sqrt{k}Z_1 + Z_2 \rangle_{\mathbb{R}}.$$

This clearly implies that  $\frac{1}{\sqrt{k}}Z + Z' = -\frac{1}{\sqrt{k}}Z + Z' = 0$ , that is  $Z = Z' = 0$ , as was to be shown.

Concerning case (II), we can assume that

$$[X_1, Z_2] = rZ_3, \quad k[X_1, Z_1] = sZ_3, \quad [X_3, Z_2] = tZ_4, \quad k[X_3, Z_1] = uZ_4,$$

where  $Z_3, Z_4$  are linearly independent and  $(s, r), (u, t) \neq (0, 0)$ . By using (9), for  $X = aX_1 + bX_2 + cX_3 + dX_4 + eZ_1 + fZ_2$  we have that

$$\begin{aligned} [X_1, X] &= cZ_1 + dZ_2 + \left(\frac{e}{k}s + fr\right)Z_3, \\ [X_2, X] &= dkZ_1 + cZ_2 + (fs + er)Z_3, \\ [X_3, X] &= -aZ_1 - bZ_2 + \left(\frac{e}{k}u + ft\right)Z_4, \\ [X_4, X] &= -bkZ_1 - aZ_2 + (fu + et)Z_4, \\ [Z_1, X] &= -\left(\frac{a}{k}s + br\right)Z_3 - \left(\frac{c}{k}u + dt\right)Z_4, \\ [Z_2, X] &= -\left(\frac{b}{k}s + ar\right)Z_3 - \left(\frac{d}{k}u + ct\right)Z_4. \end{aligned}$$

If  $a = 0$  then  $b = c = d = 0$ . We also obtain that  $e^2 = kf^2$ , since either

$$\begin{bmatrix} \frac{e}{k} & f \\ f & e \end{bmatrix} \begin{bmatrix} s \\ r \end{bmatrix} = 0 \quad \text{or} \quad \begin{bmatrix} \frac{e}{k} & f \\ f & e \end{bmatrix} \begin{bmatrix} u \\ t \end{bmatrix} = 0.$$

We do not get any plane in  $U(\mathfrak{n})$  in this way and therefore there must be an  $X \in U(\mathfrak{n})$  with  $a \neq 0$ , which implies that  $b, c, d \neq 0$  and  $a^2 = kb^2$ ,  $c^2 = kd^2$ . Thus  $[Z_1, X] = [Z_2, X] = 0$  and so  $\text{Im}(\text{ad } X) \subset \langle Z_1, Z_2 \rangle_{\mathbb{R}}$ . This implies that  $e^2 = kf^2$  and then the 3-dimensional subspace

$$\langle \sqrt{k}X_1 + X_2, \sqrt{k}X_3 + X_4, \sqrt{k}Z_1 + Z_2 \rangle_{\mathbb{R}} \subset U(\mathfrak{n}),$$

which is a contradiction, proving that case (II) is not possible and concluding the proof.  $\square$

### 3. ANOSOV DIFFEOMORPHISMS AND LIE ALGEBRAS

Anosov diffeomorphisms play an important and beautiful role in dynamics as the notion represents the most perfect kind of global hyperbolic behavior, giving examples of structurally stable dynamical systems. A diffeomorphism  $f$  of a compact differentiable manifold  $M$  is called *Anosov* if the tangent bundle  $TM$  admits a continuous invariant splitting  $TM = E^+ \oplus E^-$  such that  $df$  expands  $E^+$  and contracts  $E^-$  exponentially, that is, there exist constants  $0 < c$  and  $0 < \lambda < 1$  such that

$$\|df^n(X)\| \leq c\lambda^n\|X\|, \quad \forall X \in E^-, \quad \|df^n(Y)\| \geq c\lambda^{-n}\|Y\|, \quad \forall Y \in E^+,$$

for all  $n \in \mathbb{N}$ . The condition is independent of the Riemannian metric. Some of the other very nice properties of these special dynamical systems, all proved mainly by D. Anosov, are: the distributions  $E^+$  and  $E^-$  are completely integrable with  $C^\infty$  leaves and determine two (unique)  $f$ -invariant foliations (unstable and stable, respectively) with remarkable dynamical properties; the set of periodic points (i.e.  $f^m(p) = p$  for some  $m \in \mathbb{N}$ ) is dense in the set of those points of  $M$  such that for any neighborhood  $U$  of  $p$  there exist  $k \neq m \in \mathbb{N}$  with  $f^k(U) \cap f^m(U) \neq \emptyset$ ; the set of all Anosov diffeomorphisms form an open subset of  $\text{Diff}(M)$ .

**Example 3.1.** Let  $N$  be a real simply connected nilpotent Lie group with Lie algebra  $\mathfrak{n}$ . Let  $\varphi$  be a hyperbolic automorphism of  $N$ , that is, all the eigenvalues of its derivative  $A = (d\varphi)_e : \mathfrak{n} \mapsto \mathfrak{n}$  have absolute value different from 1. If  $\varphi(\Gamma) = \Gamma$  for some lattice  $\Gamma$  of  $N$  (i.e. a uniform discrete subgroup) then  $\varphi$  defines an Anosov diffeomorphism on the nilmanifold  $M = N/\Gamma$ , which is called an *Anosov automorphism*. The subspaces  $E^+$  and  $E^-$  are obtained by left translation of the eigenspaces of eigenvalues of  $A$  of absolute value greater than 1 and less than 1, respectively, and so the splitting is differentiable. If more in general,  $\Gamma$  is a uniform discrete subgroup of  $K \ltimes N$ , where  $K$  is any compact subgroup of  $\text{Aut}(N)$ , for which  $\varphi(\Gamma) = \Gamma$  (recall that  $\varphi$  acts on  $\text{Aut}(N)$  by conjugation), then  $\varphi$  also determines an Anosov diffeomorphism on  $M = N/\Gamma$  which is also called Anosov automorphism. In this case  $M$  is called an infranilmanifold and is finitely covered by the nilmanifold  $N/(N \cap \Gamma)$ .

In [29], S. Smale raised the problem of classifying all compact manifolds (up to homeomorphism) which admit an Anosov diffeomorphism. At this moment, the only known examples are of algebraic nature, namely Anosov automorphisms of nilmanifolds and infranilmanifolds described in the example above. It is conjectured that any Anosov diffeomorphism is topologically conjugate to an Anosov automorphism of an infranilmanifold (see [23]).

All this certainly highlights the problem of classifying all nilmanifolds which admit Anosov automorphisms, which are easily seen in correspondence with a very special class of nilpotent Lie algebras over  $\mathbb{Q}$ . Nevertheless, not too much is known on the question since it is not so easy for an automorphism of a (real) nilpotent Lie algebra being hyperbolic and unimodular at the same time.

**Definition 3.2.** A rational Lie algebra  $\mathfrak{n}^{\mathbb{Q}}$  (i.e. with structure constants in  $\mathbb{Q}$ ) of dimension  $n$  is said to be *Anosov* if it admits a *hyperbolic* automorphism  $A$  (i.e. all their eigenvalues have absolute value different from 1) which is *unimodular*, that is,  $[A]_{\beta} \in \text{GL}_n(\mathbb{Z})$  for some basis  $\beta$  of  $\mathfrak{n}^{\mathbb{Q}}$ , where  $[A]_{\beta}$  denotes the matrix of  $A$  with respect to  $\beta$ . A hyperbolic and unimodular automorphism is called an *Anosov automorphism*. We also say that a real Lie algebra is *Anosov* when it admits a rational form which is Anosov. An automorphism of a real Lie algebra  $\mathfrak{n}$  is called *Anosov* if it is hyperbolic and  $[A]_{\beta} \in \text{GL}_n(\mathbb{Z})$  for some  $\mathbb{Z}$ -basis  $\beta$  of  $\mathfrak{n}$  (i.e. with integer structure constants).

The unimodularity condition on  $A$  in the above definition is equivalent to the fact that the characteristic polynomial of  $A$  has integer coefficients and constant term equal to  $\pm 1$  (see [5]). It is well known that any Anosov Lie algebra is necessarily nilpotent, and it is easy to see that the classification of nilmanifolds which admit an Anosov automorphism is essentially equivalent to that of Anosov Lie algebras (see [18, 4, 15, 5]). If  $\mathfrak{n}$  is a rational Lie algebra, we call the real Lie algebra  $\mathfrak{n} \otimes \mathbb{R}$  the *real completion* of  $\mathfrak{n}$ .

We now give some necessary conditions a real Lie algebra has to satisfy in order to be Anosov (see [20]).

**Proposition 3.3.** *Let  $\mathfrak{n}$  be a real nilpotent Lie algebra which is Anosov. Then there exist a decomposition  $\mathfrak{n} = \mathfrak{n}_1 \oplus \dots \oplus \mathfrak{n}_r$  satisfying  $C^i(\mathfrak{n}) = \mathfrak{n}_{i+1} \oplus \dots \oplus \mathfrak{n}_r$ ,  $i = 0, \dots, r$ , and a hyperbolic  $A \in \text{Aut}(\mathfrak{n})$  such that*

- (i)  $A\mathfrak{n}_i = \mathfrak{n}_i$  for all  $i = 1, \dots, r$ .
- (ii)  $A$  is semisimple (in particular  $A$  is diagonalizable over  $\mathbb{C}$ ).

- (iii) For each  $i$ , there exists a basis  $\beta_i$  of  $\mathfrak{n}_i$  such that  $[A_i]_{\beta_i} \in SL_{n_i}(\mathbb{Z})$ , where  $n_i = \dim \mathfrak{n}_i$  and  $A_i = A|_{\mathfrak{n}_i}$ .

*Proof.* Let  $\beta$  be a  $\mathbb{Z}$ -basis of  $\mathfrak{n}$  for which there is a hyperbolic  $A \in \text{Aut}(\mathfrak{n})$  satisfying  $[A]_{\beta} \in GL_n(\mathbb{Z})$ . By using that  $\text{Aut}(\mathfrak{n})$  is a linear algebraic group, it is proved in [1, Section 2] that we can assume that  $A$  is semisimple. Thus the existence of the decomposition satisfying (i) follows from the fact that the subspaces  $C^i(\mathfrak{n})$  are  $A$ -invariant.

If  $\beta = \{X_1, \dots, X_n\}$  then the discrete (additive) subgroup

$$\mathfrak{n}^{\mathbb{Z}} = \left\{ \sum_{i=1}^n a_i X_i : a_i \in \mathbb{Z} \right\}$$

of  $\mathfrak{n}$  is closed under the Lie bracket of  $\mathfrak{n}$  and  $A$ -invariant, and  $C^i(\mathfrak{n}^{\mathbb{Z}})$  is a discrete subgroup of  $C^i(\mathfrak{n})$  of maximal rank. Since  $AC^i(\mathfrak{n}^{\mathbb{Z}}) = C^i(\mathfrak{n}^{\mathbb{Z}})$  for any  $i$  we have that  $A$  induces an invertible map

$$C^{i-1}(\mathfrak{n}^{\mathbb{Z}})/C^i(\mathfrak{n}^{\mathbb{Z}}) \mapsto C^{i-1}(\mathfrak{n}^{\mathbb{Z}})/C^i(\mathfrak{n}^{\mathbb{Z}}),$$

and it follows from  $C^i(\mathfrak{n}^{\mathbb{Z}}) \otimes \mathbb{R} = C^i(\mathfrak{n})$  that  $C^{i-1}(\mathfrak{n}^{\mathbb{Z}})/C^i(\mathfrak{n}^{\mathbb{Z}}) \simeq \mathbb{Z}^{n_i}$  is a discrete subgroup of  $C^{i-1}(\mathfrak{n})/C^i(\mathfrak{n}) \simeq \mathfrak{n}_i$  which is leaved invariant by  $A$ , proving the existence of the basis  $\beta_i$  of  $\mathfrak{n}_i$  in (iii). Recall that by considering  $A^2$  rather than  $A$  if necessary, we can assume that  $\det A_i = 1$  for all  $i$ .  $\square$

**Proposition 3.4.** *Let  $\mathfrak{n}$  be a real  $r$ -step nilpotent Lie algebra of type  $(n_1, \dots, n_r)$  (see Definition 2.2). If  $\mathfrak{n}$  is Anosov then at least one of the following is true:*

- (i)  $n_1 \geq 4$  and  $n_i \geq 2$  for all  $i = 2, \dots, r$ .
- (ii)  $n_1 = n_2 = 3$  and  $n_i \geq 2$  for all  $i = 3, \dots, r$ .

In particular,  $\dim \mathfrak{n} \geq 2r + 2$ .

*Proof.* We know from Proposition 3.3 that  $A_i \in SL_{n_i}(\mathbb{Z})$  is hyperbolic, which implies that  $n_i \geq 2$  for any  $i$ . Assuming (i) does not hold means then that  $n_1 = 3$ . If  $n_2 = 2$  and  $\{\lambda_1, \lambda_2, \lambda_3\}$  are the eigenvalues of  $A_1$  then the eigenvalues of  $A_2$  are of the form  $\lambda_i \lambda_j$ , say  $\{\lambda_1 \lambda_2, \lambda_1 \lambda_3\}$ , and hence  $\lambda_1 = \lambda_1^2 \lambda_2 \lambda_3 = 1$ , which contradicts the fact that  $A_1$  is hyperbolic. This implies that  $n_2 = 3$ .  $\square$

In [18, Question (ii)] there are examples of real Anosov Lie algebras of type  $(4, 2, \dots, 2)$  for any  $r \geq 2$ . We shall prove in Section 4, Case  $(3, 3, 2)$ , that in part (ii) of the above proposition one actually needs  $n_3 \geq 3$ . Also, we do not know of any example of type of the form  $(3, 3, \dots)$ .

Part (i) of the following proposition is essentially [1, Theorem 3]

**Proposition 3.5.** *Let  $\mathfrak{n} = \mathfrak{n}_1 \oplus \mathfrak{n}_2$  be a real 2-step nilpotent Lie algebra with  $\dim \mathfrak{n}_2 = k$ . Assume that  $\mathfrak{n}$  is Anosov and let  $\mathfrak{n}^{\mathbb{Q}}$  denote the rational form which is Anosov.*

- (i) *If  $f$  is the Pfaffian form of  $\mathfrak{n}$  then for any  $c > 0$  the region*

$$R_c = \{(x_1, \dots, x_k) \in \mathbb{R}^k : |f(x_1, \dots, x_k)| \leq c\}$$

*is unbounded.*

- (ii) *For the Pfaffian form  $f$  of  $\mathfrak{n}^{\mathbb{Q}}$  and for any  $p \in \mathbb{Z}$  the set*

$$S_p = \{(x_1, \dots, x_k) \in \mathbb{Z}^k : f(x_1, \dots, x_k) = p\}$$

*is either empty or infinite.*

*Proof.* (i) Consider  $A \in \text{Aut}(\mathfrak{n})$  satisfying all the conditions in Proposition 3.3. It follows from the proof of Proposition 2.5 and  $\det A_i = 1$  for any  $i = 1, \dots, r$  that

$$f(x_1, \dots, x_k) = f(A^t(x_1, \dots, x_k)) \quad \forall (x_1, \dots, x_k) \in \mathbb{R}^k = \mathfrak{n}_2,$$

and so  $AR_c \subset R_c$ . Assume that  $R_{c_0}$  is bounded for some  $c_0 > 0$ , by using that  $f$  is a homogeneous polynomial we get that  $R_c$  is bounded for any  $c > 0$ ; indeed,  $R_c = c^{-\frac{1}{m}} R_1$  if  $m$  is the degree of  $f$ . Now, for a sufficiently big  $c_1 > 0$  we may assume that  $R_{c_1}$  contains the basis  $\beta_2$  of  $\mathfrak{n}_2$ , but only finitely many integral linear combinations of elements in this basis can belong to the bounded region  $R_{c_1}$ . This implies that  $A^t|_{\mathfrak{n}_2}$  leave a finite set of points invariant, and since such a set contains a basis of  $\mathfrak{n}_2$  we obtain that  $(A^t)^l = I$  for some  $l \in \mathbb{N}$ . The eigenvalues of  $A$  have then to be roots of the identity, contradicting its hyperbolicity.

(ii) Analogously to the proof of part (i), we get that  $A^t S_p \subset S_p$ . If  $S_p \neq \emptyset$  and finite then for the real subspace  $W \subset \mathfrak{n}_2$  generated by  $S_p$  we have that  $A^t W \subset W$  and  $(A^t|_W)^l = I$  for some  $l \in \mathbb{N}$ , which is again a contradiction by the hyperbolicity of  $A$ .  $\square$

We now give an example of how the above proposition can be applied. Rational Lie algebras of type  $(4, 2)$  are parametrized by the set of square free numbers  $k \in \mathbb{Z}$  and their Pfaffian forms are  $f_k(x, y) = x^2 - ky^2$  (see the paragraph before Proposition 2.7). Thus the set of solutions

$$\{(x, y) \in \mathbb{Z}^2 : f_k(x, y) = 1\}$$

is infinite if and only if  $k > 1$  or  $k = 0$  (Pell equation). By Proposition 3.5, (ii), the Lie algebra  $\mathfrak{n}_k^{\mathbb{Q}}$  can never be Anosov for  $k < 0$  or  $k = 1$ . Recall that we could also discard  $\mathfrak{n}_k^{\mathbb{Q}}$ ,  $k < 0$  as a real Anosov Lie algebra by applying Proposition 3.5, (i).

It is not true in general that if a direct sum of real Lie algebras is Anosov then each of the direct summands is so, as the example  $\mathfrak{h}_3 \oplus \mathfrak{h}_3$  shows. However, we shall see next that this actually happens when one of the direct summands is (maximal) abelian.

Let  $\mathfrak{n}$  be a Lie algebra over  $K$ . An *abelian factor* of  $\mathfrak{n}$  is an abelian ideal  $\mathfrak{a}$  for which there exists an ideal  $\tilde{\mathfrak{n}}$  of  $\mathfrak{n}$  such that  $\mathfrak{n} = \tilde{\mathfrak{n}} \oplus \mathfrak{a}$  (i.e.  $[\tilde{\mathfrak{n}}, \mathfrak{a}] = 0$ ). Let  $m(\mathfrak{n})$  denote the maximum dimension over all abelian factors of  $\mathfrak{n}$ . If  $\mathfrak{z}$  is the center of  $\mathfrak{n}$  then the maximal abelian factors are precisely the linear direct complements of  $\mathfrak{z} \cap [\mathfrak{n}, \mathfrak{n}]$  in  $\mathfrak{z}$ , that is, those subspaces  $\mathfrak{a} \subset \mathfrak{z}$  such that  $\mathfrak{z} = \mathfrak{z} \cap [\mathfrak{n}, \mathfrak{n}] \oplus \mathfrak{a}$ . Therefore

$$m(\mathfrak{n}) = \dim \mathfrak{z} - \dim \mathfrak{z} \cap [\mathfrak{n}, \mathfrak{n}].$$

**Theorem 3.6.** *Let  $\mathfrak{n}$  be a rational Lie algebra with  $m(\mathfrak{n}) = r$  and let  $\mathfrak{n} = \tilde{\mathfrak{n}} \oplus \mathbb{Q}^r$  be any decomposition in ideals, that is,  $\mathbb{Q}^r$  is a maximal abelian factor of  $\mathfrak{n}$ . Then  $\mathfrak{n}$  is Anosov if and only if  $\tilde{\mathfrak{n}}$  is Anosov and  $r \geq 2$ .*

*Proof.* If  $\tilde{\mathfrak{n}}$  is Anosov and  $r \geq 2$  then we consider the automorphism  $A$  of  $\mathfrak{n}$  defined on  $\tilde{\mathfrak{n}}$  as an Anosov automorphism of  $\tilde{\mathfrak{n}}$  and on  $\mathbb{Q}^r$  as any hyperbolic matrix in  $GL_r(\mathbb{Z})$ . Thus  $A$  is an Anosov automorphism of  $\mathfrak{n}$ .

Conversely, let  $A$  be an Anosov automorphism of  $\mathfrak{n}$ . As in the proof of Proposition 3.3 we may assume that  $A$  is semisimple and consider the discrete (additive) subgroup

$$\mathfrak{n}^{\mathbb{Z}} = \left\{ \sum_{i=1}^n a_i X_i, a_i \in \mathbb{Z} \right\}$$

which is  $A$ -invariant. Since the center  $\mathfrak{z}$  of  $\mathfrak{n}$  and  $\mathfrak{z}_1 = \mathfrak{z} \cap [\mathfrak{n}, \mathfrak{n}]$  are both leaved invariant by  $A$ , there exist  $A$ -invariant subspaces  $V$  and  $\mathfrak{a} \subset \mathfrak{z}$  such that

$$\mathfrak{n} = V \oplus \mathfrak{z} = V \oplus \mathfrak{z}_1 \oplus \mathfrak{a}.$$

Thus  $\mathfrak{a}$  is a maximal abelian factor,  $\dim \mathfrak{a} = r$  and  $A$  has the form

$$A = \begin{bmatrix} A_1 & & \\ & A_2 & \\ & & A_3 \end{bmatrix}, \quad A_1 = A|_V, \quad A_2 = A|_{\mathfrak{z}_1}, \quad A_3 = A|_{\mathfrak{a}}.$$

The subgroup  $\mathfrak{z}(\mathfrak{n}_{\mathbb{Z}}) = \{X \in \mathfrak{n}_{\mathbb{Z}} : [X, Y] = 0 \ \forall Y \in \mathfrak{n}_{\mathbb{Z}}\}$  is also  $A$ -invariant and it is a lattice of  $\mathfrak{z}$  (i.e. a discrete subgroup of maximal rank) since for any  $Z \in \mathfrak{z}$  there exist  $k \in \mathbb{Z}$  such that  $kZ \in \mathfrak{z}(\mathfrak{n}_{\mathbb{Z}})$  and  $Z = \frac{1}{k}(kZ)$ , that is,  $\mathfrak{z}(\mathfrak{n}_{\mathbb{Z}}) \otimes \mathbb{Q} = \mathfrak{z}$ . Since  $\mathfrak{n}_{\mathbb{Z}}/\mathfrak{z}(\mathfrak{n}_{\mathbb{Z}})$  is  $A$ -invariant and  $(\mathfrak{n}_{\mathbb{Z}}/\mathfrak{z}(\mathfrak{n}_{\mathbb{Z}})) \otimes \mathbb{Q} \simeq V$  we get that  $A_1$  is unimodular. Analogously,  $A_2$  and  $A_3$  are unimodular since  $\mathfrak{z}_1(\mathbb{Z}) = \mathfrak{z}(\mathfrak{n}_{\mathbb{Z}}) \cap [\mathfrak{n}_{\mathbb{Z}}, \mathfrak{n}_{\mathbb{Z}}]$  and  $\mathfrak{z}(\mathfrak{n}_{\mathbb{Z}})/\mathfrak{z}_1(\mathbb{Z})$  are also discrete subgroups of maximal rank of  $\mathfrak{z}_1$  and  $\mathfrak{z}/\mathfrak{z}_1 \simeq \mathfrak{a}$ , respectively.

The hyperbolicity of  $A$  guaranties the one of  $A_1, A_2$  and  $A_3$  and so  $\tilde{\mathfrak{n}} \simeq V \oplus \mathfrak{z}_1$  is Anosov and  $\dim \mathfrak{a} \geq 2$ , as we wanted to show.  $\square$

To finish this section, we give a simple procedure to construct explicit examples of Anosov Lie algebras. This result is a generalization of [18, Theorem 3.1] proposed by F. Grunewald.

A Lie algebra  $\mathfrak{n}$  over  $K$  is said to be *graded* (over  $\mathbb{N}$ ) if there exist  $K$ -subspaces  $\mathfrak{n}_i$  of  $\mathfrak{n}$  such that

$$\mathfrak{n} = \mathfrak{n}_1 \oplus \mathfrak{n}_2 \oplus \dots \oplus \mathfrak{n}_k \quad \text{and} \quad [\mathfrak{n}_i, \mathfrak{n}_j] \subset \mathfrak{n}_{i+j}.$$

Equivalently,  $\mathfrak{n}$  is graded when there are nonzero  $K$ -subspaces  $\mathfrak{n}_{d_1}, \dots, \mathfrak{n}_{d_r}$ ,  $d_1 < \dots < d_r$ , such that  $\mathfrak{n} = \mathfrak{n}_{d_1} \oplus \dots \oplus \mathfrak{n}_{d_r}$  and if  $0 \neq [\mathfrak{n}_{d_i}, \mathfrak{n}_{d_j}]$  then  $d_i + d_j = d_k$  for some  $k$  and  $[\mathfrak{n}_{d_i}, \mathfrak{n}_{d_j}] \subset \mathfrak{n}_{d_k}$ . Recall that any graded Lie algebra is necessarily nilpotent.

**Theorem 3.7.** *Let  $\mathfrak{n}^{\mathbb{Q}}$  be a graded rational Lie algebra, and consider the direct sum  $\tilde{\mathfrak{n}}^{\mathbb{Q}} = \mathfrak{n}^{\mathbb{Q}} \oplus \dots \oplus \mathfrak{n}^{\mathbb{Q}}$  ( $s$  times,  $s \geq 2$ ). Then the real Lie algebra  $\tilde{\mathfrak{n}} = \tilde{\mathfrak{n}}^{\mathbb{Q}} \otimes \mathbb{R}$  is Anosov. In other words, if  $\mathfrak{n}$  is a graded real Lie algebra admitting a rational form, then  $\tilde{\mathfrak{n}} = \mathfrak{n} \oplus \dots \oplus \mathfrak{n}$  ( $s$ -times,  $s \geq 2$ ) is Anosov.*

*Proof.* Let  $\{X_1, \dots, X_n\}$  be a  $\mathbb{Z}$ -basis of  $\mathfrak{n}^{\mathbb{Q}}$  compatible with the gradation  $\mathfrak{n}^{\mathbb{Q}} = \mathfrak{n}_{d_1}^{\mathbb{Q}} \oplus \dots \oplus \mathfrak{n}_{d_r}^{\mathbb{Q}}$ , that is, a basis with integer structure constants and such that each  $X_i \in \mathfrak{n}_{d_j}^{\mathbb{Q}}$  for some  $j$ . We will denote this basis by  $\{X_{l1}, \dots, X_{ln}\}$  when we need to make clear that it is a basis of the  $l$ -th copy of  $\mathfrak{n}^{\mathbb{Q}}$  in  $\tilde{\mathfrak{n}}^{\mathbb{Q}}$ , so the Lie bracket of  $\tilde{\mathfrak{n}}^{\mathbb{Q}}$  is given by  $[X_{li}, X_{l'j}] = 0$  for all  $l \neq l'$ , and for any  $l = 1, \dots, s$

$$(11) \quad [X_{li}, X_{lj}] = \sum_{k=1}^n m_{ij}^k X_{lk}, \quad m_{ij}^k \in \mathbb{Z}.$$

Every nonzero  $\lambda \in \mathbb{R}$  defines an automorphism  $A_{\lambda}$  of  $\mathfrak{n}^{\mathbb{Q}} \otimes \mathbb{R}$  by

$$A_{\lambda}|_{\mathfrak{n}_{d_i}^{\mathbb{Q}} \otimes \mathbb{R}} = \lambda^{d_i} I.$$

Let  $B$  be a matrix in  $GL_s(\mathbb{Z})$  with eigenvalues  $\lambda_1, \dots, \lambda_s$  and assume that all of them are real numbers different from  $\pm 1$  (we are using here that  $s \geq 2$ ). This determines an automorphism  $A$  of  $\tilde{\mathfrak{n}}$  in the following way:  $A$  leaves the decomposition  $\tilde{\mathfrak{n}}^{\mathbb{Q}} = (\mathfrak{n}^{\mathbb{Q}} \otimes \mathbb{R}) \oplus \dots \oplus (\mathfrak{n}^{\mathbb{Q}} \otimes \mathbb{R})$  invariant and on the  $l$ -th copy of  $\mathfrak{n}^{\mathbb{Q}} \otimes \mathbb{R}$  coincides with  $A_{\lambda_l}$ .

Consider the new basis of  $\tilde{\mathfrak{n}}$  defined by

$$\begin{aligned} \beta = \{ & X_{11} + X_{21} + \dots + X_{s1}, \lambda_1 X_{11} + \lambda_2 X_{21} + \dots + \lambda_s X_{s1}, \dots, \\ & \lambda_1^{s-1} X_{11} + \lambda_2^{s-1} X_{21} + \dots + \lambda_s^{s-1} X_{s1}, \dots, X_{1n} + X_{2n} + \dots + X_{sn}, \\ & \lambda_1 X_{1n} + \lambda_2 X_{2n} + \dots + \lambda_s X_{sn}, \dots, \lambda_1^{s-1} X_{1n} + \lambda_2^{s-1} X_{2n} + \dots + \lambda_s^{s-1} X_{sn} \}. \end{aligned}$$

In order to prove that  $\beta$  is also a  $\mathbb{Z}$ -basis we take two generic elements of it, say  $X = \lambda_1^t X_{1i} + \lambda_2^t X_{2i} + \dots + \lambda_s^t X_{si}$  and  $Y = \lambda_1^u X_{1j} + \lambda_2^u X_{2j} + \dots + \lambda_s^u X_{sj}$  for some  $0 \leq t, u \leq s-1$  and  $1 \leq i, j \leq n$ . Since the  $\lambda_l$ 's are all roots of the characteristic polynomial  $f(x) = a_0 + a_1 x + \dots + a_{s-1} x^{s-1} + x^s$  of  $B$  (with  $a_i \in \mathbb{Z}$  and  $a_0 = \pm 1$ ), there exist  $b_0, \dots, b_{s-1} \in \mathbb{Z}$  (independent from  $l$ ) such that  $\lambda_l^{t+u} = b_0 + b_1 \lambda_l + \dots + b_{s-1} \lambda_l^{s-1}$  for any  $l = 1, \dots, s$ . Now, by using (11) we obtain that

$$\begin{aligned} [X, Y] &= \lambda_1^{t+u} [X_{1i}, X_{1j}] + \dots + \lambda_s^{t+u} [X_{si}, X_{sj}] \\ &= \sum_{k=1}^n m_{ij}^k \lambda_1^{t+u} X_{1k} + \dots + \sum_{k=1}^n m_{ij}^k \lambda_s^{t+u} X_{sk} \\ &= \sum_{k=1}^n m_{ij}^k b_0 (X_{1k} + \dots + X_{sk}) + \sum_{k=1}^n m_{ij}^k b_1 (\lambda_1 X_{1k} + \dots + \lambda_s X_{sk}) \\ &\quad + \dots + \sum_{k=1}^n m_{ij}^k b_{s-1} (\lambda_1^{s-1} X_{1k} + \dots + \lambda_s^{s-1} X_{sk}), \end{aligned}$$

showing that  $\beta$  is also a  $\mathbb{Z}$ -basis of  $\tilde{\mathfrak{n}}$ . Thus the linear combinations over  $\mathbb{Q}$  of  $\beta$  determine a rational form of  $\tilde{\mathfrak{n}}$ , denoted by  $\mathfrak{n}_\beta^\mathbb{Q}$ , which will be now showed to be Anosov. Indeed, it is easy to see that, written in terms of  $\beta$ , the hyperbolic automorphism  $A$  of  $\tilde{\mathfrak{n}}$  has the form

$$[A]_\beta = \begin{bmatrix} B' & & \\ & \ddots & \\ & & B' \end{bmatrix} \in GL_{ns}(\mathbb{Z}),$$

where

$$B' = \begin{bmatrix} 0 & 0 & & -a_0 \\ 1 & 0 & & -a_1 \\ 0 & 1 & & \\ & & \ddots & \\ 0 & 0 & & 1 & -a_{s-1} \end{bmatrix} \in GL_s(\mathbb{Z})$$

is the rational form of the matrix  $B$ , concluding the proof of the theorem.  $\square$

Different choices of matrices  $B$  can eventually give non-isomorphic Anosov rational forms of  $\tilde{\mathfrak{n}}$ , as in the case  $\tilde{\mathfrak{n}} = \mathfrak{h}_3 \oplus \mathfrak{h}_3$  and  $\tilde{\mathfrak{n}} = \mathfrak{l}_4 \oplus \mathfrak{l}_4$  (see also [30]). Recall that two-step nilpotent Lie algebras are graded, so Theorem 3.7 shows that a reasonable classification of Anosov Lie algebras up to isomorphism is far from being feasible, not only in the rational case but even in the real case (see [18] for further information).

**Remark 3.8.** The only explicit examples of real Anosov Lie algebras in the literature so far which are not covered by Theorem 3.7 are the following: the free  $k$ -step nilpotent Lie algebras on  $n$  generators with  $k < n$  (see [4], and also [7, 5] for a different approach); certain  $k$ -step nilpotent Lie algebras of dimension  $d + \binom{d}{2} + \dots + \binom{d}{k}$  with  $d \geq k^2$  (see [9]); the 2-step nilpotent Lie algebra of type  $(d, \binom{d}{2} - 1)$  with center of codimension  $d$  for  $d \geq 5$  (see [6]); and the Lie algebra  $\mathfrak{g}$  (see [18]). Thus  $\mathfrak{h}$  is the only new example over  $\mathbb{R}$  obtained in the classification in dimension  $\leq 8$  (see Table 3). For the known examples of infranilmanifolds which are not nilmanifolds and admit Anosov automorphisms we refer to [28, 24, 21].

The *signature* of an Anosov diffeomorphism is the pair of natural numbers  $\{p, q\} = \{\dim E^+, \dim E^-\}$ . It is known that signature  $\{1, n-1\}$  is only possible for torus (and their finitely covered spaces: compact flat manifolds) (see [10]).

If  $\dim \mathfrak{n}^{\mathbb{Q}} = n$  then the signature of the Anosov automorphism of  $\tilde{\mathfrak{n}}^{\mathbb{Q}} \otimes \mathbb{R}$  ( $\tilde{\mathfrak{n}}^{\mathbb{Q}} = \mathfrak{n}^{\mathbb{Q}} \oplus \dots \oplus \mathfrak{n}^{\mathbb{Q}}$ ,  $s$  times) in the proof of Theorem 3.7 is  $\{np', nq'\}$ ,  $p' + q' = s$ , where  $p', q'$  are the numbers of eigenvalues of  $B \in GL_s(\mathbb{Z})$  having module greater and smaller than 1, respectively. In the nonabelian case  $n$  is necessarily  $\geq 3$  and so the signature  $\{2, q\}$  is not allowed for this construction. We do not actually of any nonabelian example of signature  $\{2, q\}$ . We may choose  $\{p', q'\} = \{1, s-1\}$  and  $\mathfrak{n}^{\mathbb{Q}} \otimes \mathbb{R} = \mathfrak{h}_3$  in order to obtain signature  $\{3, 3(s-1)\}$  for any  $s \geq 2$ .

#### 4. CLASSIFICATION OF REAL ANOSOV LIE ALGEBRAS

We will find in this section all the real Anosov Lie algebras of dimension  $\leq 8$ . Our start point is Proposition 3.4, which implies that a nonabelian one has to be of dimension  $\geq 6$  and gives only a few possibilities for the types in each dimension 6, 7 and 8.

We use Proposition 3.3 to make a few observations on the eigenvalues of an Anosov automorphism, which are necessarily algebraic integers. An overview on several basic properties of algebraic numbers is given in the Appendix.

**Lemma 4.1.** *Let  $\mathfrak{n}$  be a real nilpotent Lie algebra which is Anosov, and let  $A$  and  $\mathfrak{n} = \mathfrak{n}_1 \oplus \mathfrak{n}_2 \oplus \dots \oplus \mathfrak{n}_r$  be as in Proposition 3.3. If  $A_i = A|_{\mathfrak{n}_i}$  then the corresponding eigenvalues  $\lambda_1, \dots, \lambda_{n_i}$ , are algebraic units such that  $1 < \deg \lambda_i \leq n_i$  and  $\lambda_1 \dots \lambda_{n_i} = 1$ .*

This follows from the fact that  $[A_i]_{\beta_i} \in SL_{n_i}(\mathbb{Z})$  and so its characteristic polynomial  $p_{A_i}(x) \in \mathbb{Z}[x]$  is a monic polynomial with constant coefficient  $a_0 = (-1)^n \det A_i = \pm 1$ , satisfying  $p_{A_i}(\lambda_j) = 0$  for all  $j = 1, \dots, n_i$ .

Concerning the degree, it is clear that  $\deg \lambda_j \leq n_i$  for all  $j$  and if  $\deg \lambda_j = 1$  then  $\lambda_j \in \mathbb{Q}$ , is a positive unit and therefore  $\lambda_j = 1$ , contradicting the fact that  $A_i$  is hyperbolic.

In the following,  $\mathfrak{n}$ ,  $A$ ,  $A_i$  and  $\mathfrak{n}_i$  will be as in the previous lemma. In order to be able of working with eigenvectors, we will always consider the complex Lie algebra  $\mathfrak{n}_{\mathbb{C}} = \mathfrak{n} \otimes \mathbb{C}$  and its decomposition  $\mathfrak{n}_{\mathbb{C}} = (\mathfrak{n}_1)_{\mathbb{C}} \oplus \dots \oplus (\mathfrak{n}_r)_{\mathbb{C}}$ , where  $(\mathfrak{n}_i)_{\mathbb{C}} = \mathfrak{n}_i \otimes \mathbb{C}$ . In the light of Theorem 3.6, we will always assume that  $\mathfrak{n}$  has no abelian factor. We now fix more notation that will be used in the rest of this section. For simplicity, assume that  $\mathfrak{n}$  is a 2-step nilpotent Lie algebra. According to Proposition 3.3, there exist

$$\beta_1 = \{X_1, X_2, \dots, X_{n_1}\} \quad \text{and} \quad \beta_2 = \{Z_1, Z_2, \dots, Z_{n_2}\},$$

basis of eigenvectors of  $\mathfrak{n}_1)_\mathbb{C}$  and  $(\mathfrak{n}_2)_\mathbb{C}$  for  $A_1$  and  $A_2$ , respectively. Let  $\lambda_1, \dots, \lambda_{n_1}$  and  $\mu_1, \dots, \mu_{n_2}$  be the corresponding eigenvalues. This notation will be used throughout all the classification. The absence of abelian factor implies that  $[\mathfrak{n}_1, \mathfrak{n}_1] = \mathfrak{n}_2$  and hence we may assume that for each  $Z_i$  there exist  $X_j$  and  $X_l$  such that  $Z_i = [X_j, X_l]$ . On the other hand, for each  $j, l$ , there exist scalars  $a_k^{j,l} \in \mathbb{C}$  such that  $[X_j, X_l] = \sum a_k^{j,l} Z_k$ . Since  $\{Z_k\}$  are linearly independent, for each  $k$  we obtain

$$(12) \quad \lambda_j \lambda_l a_k^{j,l} = \mu_k a_k^{j,l}.$$

Hence, if  $a_k^{j,l} \neq 0$ ,  $\mu_k = \lambda_j \lambda_l$ , and therefore, if  $a_k^{j,l} \neq 0 \neq a_{k'}^{j,l}$ ,  $\mu_k = \mu_{k'}$ . In particular, if  $n_2 = 2$ , since  $\mu_1 \neq \mu_2$ , for each  $j, l$ , there exist a unique  $k$  such that  $[X_j, X_l] = a_k Z_k$ . If it is so, by (12),  $\lambda_j \lambda_l = \mu_k$ . When  $n_2 = 3$  the same property holds. Indeed,  $\mu_i \neq \mu_j$  for all  $i \neq j$  since  $\text{dgr } \mu_i > 1$  for all  $i$ .

We are going to consider all the possible coefficients  $a_k^{j,l}$ 's only in the cases when the classification actually leads to a possible Anosov Lie algebra.

### Dimension $\leq 6$

Anosov Lie algebras of dimension  $\leq 6$  has already been classified in [20] and [3]. We give an alternative proof here in order to illustrate our approach.

Proposition 3.4 gives us the following possibilities for the types of a real Anosov Lie algebra without an abelian factor:  $(3, 3)$  and  $(4, 2)$ .

**Case  $(3, 3)$ .** The only real (resp. rational) Lie algebra of type  $(3, 3)$  is the free 2-step nilpotent Lie algebra on 3 generators  $\mathfrak{f}_3$  (resp.  $\mathfrak{f}_3^\mathbb{Q}$ ), which is proved to be Anosov in [4] and [5, 20].

**Case  $(4, 2)$ .** Let  $\mathfrak{n}$  be a real nilpotent Lie algebra of type  $(4, 2)$ , admitting a hyperbolic automorphism  $A$  as in Proposition 3.3. If  $\{X_1, \dots, X_4\}$  is a basis of  $(\mathfrak{n}_1)_\mathbb{C}$  of eigenvectors of  $A_1$  with corresponding eigenvalues  $\lambda_1, \dots, \lambda_4$ , then without any loss of generality we may assume that we are in one of the following cases:

- (a)  $[X_1, X_2] = Z_1, \quad [X_1, X_3] = Z_2,$
- (b)  $[X_1, X_2] = Z_1, \quad [X_3, X_4] = Z_2.$

In the first situation, (a) implies that  $\lambda_1^2 \lambda_2 \lambda_3 = 1$ , and since  $\det A_1 = \lambda_1 \lambda_2 \lambda_3 \lambda_4 = 1$ , we obtain that  $\lambda_1 = \lambda_4$ . From this it is easy to see that  $\text{dgr } \lambda_1 = \text{dgr } \lambda_4 = 2$  and moreover,  $\lambda_2 = \lambda_3 = \lambda_1^{-1}$  (see Appendix). Therefore, we get to the contradiction  $\mu_1 = \mu_2 = 1$ .

Concerning (b), we may assume that there is no more Lie brackets among the  $\{X_i\}$  since otherwise we will be in situation (a), and thus  $\mathfrak{n}_\mathbb{C} \simeq (\mathfrak{h}_3 \oplus \mathfrak{h}_3)_\mathbb{C}$ . This Lie algebra has two real forms:  $\mathfrak{h}_3 \oplus \mathfrak{h}_3$  and  $\mathfrak{n}_{-1}^\mathbb{Q} \otimes \mathbb{R}$  (see paragraph before Proposition 2.7). The Lie algebra  $\mathfrak{n}_{-1}^\mathbb{Q} \otimes \mathbb{R}$  can not be Anosov by Proposition 3.5, (i), and  $\mathfrak{h}_3 \oplus \mathfrak{h}_3$  is Anosov by Theorem 3.7.

### Dimension 7

According to Proposition 3.4, if  $\mathfrak{n}$  is a 7-dimensional real Anosov Lie algebra of type  $(n_1, n_2, \dots, n_r)$ , then  $r = 2$  and  $\mathfrak{n}$  is either of type  $(4, 3)$  or  $(5, 2)$ . We shall prove that there is no Anosov Lie algebras of any of these types.

**Case  $(4, 3)$ .** It is easy to see that the eigenvalues of  $A_2$  are three pairs of the form  $\lambda_i \lambda_j$ , so without any loss of generality we can assume that two of them are  $\lambda_1 \lambda_2$

and  $\lambda_1\lambda_3$ . There are four possibilities for the third eigenvalue of  $A_2$ , and by using that  $\det A_1 = 1$  and  $\det A_2 = 1$  we get to a contradiction in all the cases as follows:

- (i)  $\lambda_1\lambda_2.\lambda_1\lambda_3.\lambda_1\lambda_4 = 1$ , then  $\lambda_1^2 = 1$  contradicting the hyperbolicity of  $A_1$ .
- (ii)  $\lambda_1\lambda_2.\lambda_1\lambda_3.\lambda_2\lambda_3 = 1$  implies that  $\lambda_4^2 = 1$ , but then  $A_1$  is not hyperbolic.
- (iii)  $\lambda_1\lambda_2.\lambda_1\lambda_3.\lambda_2\lambda_4 = 1$ , then  $\lambda_1\lambda_2 = 1$  and so  $A_2$  would not be hyperbolic.
- (iv)  $\lambda_1\lambda_2.\lambda_1\lambda_3.\lambda_3\lambda_4 = 1$ , so  $\lambda_1\lambda_3 = 1$  contradicting the hyperbolicity of  $A_2$ .

**Case (5, 2).** Let  $\mathfrak{n}$  be a real nilpotent Lie algebra of type (5, 2), admitting a hyperbolic automorphism  $A$  as in Proposition 3.3. If  $\lambda_1, \dots, \lambda_5$ , are the eigenvalues of  $A_1$  we can either have

- (i)  $\lambda_i \neq \lambda_j$ ,  $1 \leq i, j \leq 5$ , or
- (ii) after reordering if necessary,  $\lambda_1 = \lambda_2$ .

Note that in (ii),  $\lambda_1 = \lambda_2$  implies that  $2 \leq 2 \deg \lambda_1 \leq 5$  and therefore  $\deg \lambda_1 = \deg \lambda_2 = 2$ . From this it is easy to see that there exist  $i \in \{3, 4, 5\}$  such that  $\deg \lambda_i = 1$ , contradicting the hyperbolicity of  $A_1$ . Therefore, we assume (i).

On the other hand, since  $\dim \mathfrak{n}_2 = 2$ , we have two linearly independent Lie brackets among the  $\{X_i\}$ , the basis of  $(\mathfrak{n}_1)_{\mathbb{C}}$  of eigenvectors of  $A_1$ . Note that if they come from disjoint pairs of  $X_i$ , since  $\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5 = 1$ , it is clear that we would have  $\lambda_i = 1$  for some  $1 \leq i \leq 5$ . Therefore, without any loss of generality we can only consider the case when we have at least the following non trivial Lie brackets:

$$(13) \quad [X_1, X_2] = Z_1, \quad [X_1, X_3] = Z_2.$$

In the following we will show that either  $X_4$  or  $X_5$  are in the center of  $\mathfrak{n}$ , which would generate an abelian factor and hence a contradiction. From (13) we have that

$$(14) \quad \lambda_1^2\lambda_2\lambda_3 = 1, \quad \text{and then} \quad \lambda_4\lambda_5 = \lambda_1.$$

Therefore,  $[X_4, X_5] = 0$  because both of the assumptions  $[X_4, X_5] = cZ_1$  and  $[X_4, X_5] = cZ_2$  with  $c \neq 0$  leads to the contradictions  $\lambda_2 = 1$  and  $\lambda_3 = 1$ , respectively. Also, if  $[X_4, X_j] \neq 0$  and  $[X_5, X_k] \neq 0$  for some  $1 \leq j, k \leq 3$ , it follows from (i) that we only have the following possibilities:

$$\begin{array}{ll} [X_4, X_3] = cZ_1 & \text{and} \quad [X_5, X_2] = dZ_2, \quad \text{or} \\ [X_5, X_3] = cZ_1 & \text{and} \quad [X_4, X_2] = dZ_2, \end{array}$$

$(c, d \neq 0)$  which are clearly equivalent. Let us suppose then the first one, and hence

$$\text{I. } \lambda_3\lambda_4 = \lambda_1\lambda_2 \quad \text{and} \quad \text{II. } \lambda_5\lambda_2 = \lambda_1\lambda_3.$$

From I, and using that  $\lambda_4\lambda_5 = \lambda_1$  we obtain  $\lambda_3 = \lambda_2\lambda_5$ . Therefore by II,  $\lambda_1 = 1$  which is a contradiction and then  $[X_4, X_j] = 0$  for all  $j$  or  $[X_5, X_k] = 0$  for all  $k$  as we wanted to show.

### Dimension 8

In this case, Proposition 3.4 gives us the following possibilities for the types of a real Anosov Lie algebra without an abelian factor: (4, 4), (5, 3), (6, 2), (3, 3, 2) and (4, 2, 2). Among all this Lie algebras we will show that there is, up to isomorphism, only three which are Anosov. One is of type (4, 2, 2), one of type (6, 2) and one of type (4, 4). The first one is an example of the construction given in [18] and

Theorem 3.7, and the second one is isomorphic to [18, Example 3.3]. The last one is a new example.

**Case (4, 4).** We will show that there is only one real Anosov Lie algebra of this type. We first note that there is only  $\binom{4}{2} = 6$  possible linearly independent brackets among the  $\{X_i\}$  and since  $\dim[\mathfrak{n}, \mathfrak{n}] = 4$ , at most two of them can be zero. Therefore, without any loss of generality, we can just consider the following two cases:

$$(15) \quad [X_1, X_3] = Z_1, \quad [X_2, X_4] = Z_2, \quad [X_2, X_3] = Z_3, \quad [X_1, X_4] = Z_4,$$

that is, the possible zero brackets corresponds to disjoint pairs of  $\{X_i\}$  (namely  $\{X_1, X_2\}$  and  $\{X_3, X_4\}$ ); and the other case is

$$(16) \quad [X_1, X_4] = Z_1, \quad [X_2, X_4] = Z_2, \quad [X_3, X_4] = Z_3, \quad [X_2, X_3] = Z_4,$$

corresponding to the case of non disjoint pairs,  $\{X_1, X_2\}$  and  $\{X_1, X_3\}$ .

However, the second case is not possible because we would have

$$\text{I) } \lambda_1 \lambda_2 \lambda_3 \lambda_4 = 1 \quad \text{and} \quad \text{II) } \lambda_1 \lambda_2^2 \lambda_3^2 \lambda_4^3 = 1.$$

It follows that  $\lambda_2 \lambda_3 = \lambda_4^{-2}$  and replacing this in I) we get  $\lambda_1 = \lambda_4$ . This implies that the  $\lambda_i$ 's have all degree two, and  $\lambda_2 = \lambda_3 = \lambda_4^{-1}$  (see Appendix). Hence  $\mu_3 = \lambda_3 \lambda_4 = 1$ , contradicting the hyperbolicity of  $A_2$ .

Concerning case (15), if we assume that  $[X_1, X_2] = 0$  and  $[X_3, X_4] = 0$

$$A = \begin{bmatrix} A_1 & \\ & A_2 \end{bmatrix}, \quad \text{where} \quad A_1 = \begin{bmatrix} \lambda & & \\ & \lambda^{-1} & \\ & & \lambda^2 \\ & & & \lambda^{-2} \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} \lambda^3 & & \\ & \lambda^{-3} & \\ & & \lambda \\ & & & \lambda^{-1} \end{bmatrix}$$

is an automorphism of  $\mathfrak{n}$  for any  $\lambda \in \mathbb{R}^*$ . If  $\lambda \in \mathbb{R}^*$  is an algebraic integer such that  $\lambda + \lambda^{-1} = 2a$ ,  $a \in \mathbb{Z}$ ,  $a \geq 2$ , then it is easy to check that

$$(17) \quad \beta = \left\{ X_1 + X_2, (a^2 - 1)^{\frac{1}{2}}(X_1 - X_2), X_3 + X_4, (a^2 - 1)^{\frac{1}{2}}(X_3 - X_4), \right. \\ \left. Z_1 + Z_2, (a^2 - 1)^{\frac{1}{2}}(Z_1 - Z_2), Z_3 + Z_4, (a^2 - 1)^{\frac{1}{2}}(Z_3 - Z_4) \right\}$$

is a  $\mathbb{Z}$ -basis of  $\mathfrak{n}$ . Moreover, if  $B = \begin{bmatrix} a & a^2 - 1 \\ 1 & a \end{bmatrix}$ , then the matrix of  $A$  in terms of the basis  $\beta$  is given by

$$[A]_\beta = \begin{bmatrix} B & & & \\ & B^2 & & \\ & & B^3 & \\ & & & B \end{bmatrix} \in SL(8, \mathbb{Z}),$$

showing that  $\mathfrak{n}$  is Anosov. Recall that this  $\mathfrak{n}$  is isomorphic to the Lie algebra  $\mathfrak{h}$  given in Example 2.12.

It follows from Scheuneman duality that there is only one more real form of  $\mathfrak{h}_{\mathbb{C}}$ , namely, the dual of the Lie algebra  $\mathfrak{n}_{-1}^{\mathbb{Q}} \otimes \mathbb{R}$  of type (4, 2) ( $\mathfrak{h}$  is dual of  $\mathfrak{h}_3 \oplus \mathfrak{h}_3$ ). The fact that such a Lie algebra is not Anosov will be proved in Section 5, Case  $\mathfrak{h}$ .

We will now show that if we add any more nonzero brackets in case (15), then the new Lie algebra  $\tilde{\mathfrak{n}}$  does not admit a hyperbolic automorphism any longer. Suppose then that

$$0 \neq [X_1, X_2] = a_1 Z_1 + a_2 Z_2 + a_3 Z_3 + a_4 Z_4.$$

As we have already pointed out at the beginning of this classification, since  $A$  is an automorphism and  $Z_i$  are linearly independent, it follows that if  $a_j \neq 0$  then  $\lambda_1 \lambda_2 = \mu_j$ . Therefore, at most two of them can be non zero.

If  $[X_1, X_2] = a_j Z_j$  then we can change  $Z_j$  by  $\tilde{Z}_j = a_j Z_j$  and the corresponding bracket in (15) by  $[X_1, X_2]$  and we will be in the conditions of case (16).

If  $[X_1, X_2] = a_j Z_j + a_k Z_k$ ,  $a_j, a_k \neq 0$ , then we have that  $\lambda_1 \lambda_2 = \mu_j = \mu_k$ . One can check that for all the choices of  $j, k$  we obtain  $\lambda_i = \lambda_r = \lambda_s$  for some  $1 \leq i, r, s \leq 4$  which is not possible because it implies that  $2 \leq 3 \deg \lambda_i \leq 4$  and then  $\deg \lambda_i = 1$ .

Hence we get  $[X_1, X_2] = 0$  and by using the same argument we also obtain  $[X_3, X_4] = 0$  as we wanted to show.

We also note that for any choice of nonzero scalars  $a, b, c, d$ , the Lie algebra  $\tilde{\mathfrak{n}}$  given by

$$[X_1, X_3] = aZ_1 \quad [X_2, X_4] = bZ_2 \quad [X_2, X_3] = cZ_3 \quad [X_1, X_4] = dZ_4,$$

is isomorphic to  $\mathfrak{n}$ .

**Case (5, 3).** We shall prove that there are no Lie algebras of this type with no abelian factor admitting a hyperbolic automorphism.

Suppose that  $A$  is as in Proposition 3.3. Hence as we have already pointed out, the eigenvalues of  $A_1$ ,  $\lambda_1, \dots, \lambda_5$  are algebraic integers with  $2 \leq \deg \lambda_j \leq 5$  for all  $1 \leq j \leq 5$ . As we have seen in case (5, 2), we can assume that  $\lambda_i \neq \lambda_j$  for all  $i \neq j$  since otherwise we will have that there exists  $k$  with  $\deg \lambda_k = 1$ , contradicting the hyperbolicity of  $A_1$ . In this situation it is easy to see that

$$(18) \quad \text{if } \#(\{X_i, X_j\} \cap \{X_k, X_l\}) = 1 \quad \text{then} \quad [X_i, X_j] \notin \mathbb{C}[X_k, X_l].$$

Moreover, since  $2 \leq \deg \mu_k \leq 3$  we have that  $\mu_k \neq \mu_l$  for all  $1 \leq k \neq l \leq 3$  and then for all  $i, j$  there exist  $k$  such that  $[X_i, X_j] \in \mathbb{C}Z_k$ .

On the other hand, it is clear that we can split the set of Lie algebras of this type according to the following condition:

$$(19) \quad \begin{array}{l} \text{There are two disjoint pairs of } \{X_i\} \text{ such that the corresponding} \\ \text{Lie brackets are linearly independent.} \end{array}$$

Note that if  $\mathfrak{n}$  does not satisfy this condition, we will have that

$$(20) \quad \{X_i, X_j\} \cap \{X_l, X_k\} = \emptyset \quad \Rightarrow \quad [X_i, X_j] \in \mathbb{C}[X_l, X_k].$$

If (20) holds, we can assume without any loss of generality that

$$(21) \quad [X_1, X_2] = Z_1 \quad [X_1, X_3] = Z_2,$$

and for  $Z_3$  we have two possibilities

$$\text{a) } [X_1, X_4] = Z_3, \quad \text{b) } [X_2, X_3] = Z_3.$$

We will now show that any of this assumptions leads to a contradiction.

Concerning a), we have that  $[X_5, X_k] \neq 0$  for some  $1 \leq k \leq 4$ , but since  $\lambda_i \neq \lambda_j$ , when  $i \neq j$  it is clear that  $k \neq 1$ . We can assume then that  $k = 2$ , since every other choice (i.e.  $k = 3, 4$ ) is entirely analogous. Now, since  $\{5, 2\} \cap \{1, 3\} = \emptyset$ , by (20) we have that  $[X_5, X_2] \in \mathbb{C}Z_2$ , and analogously,  $\{5, 2\} \cap \{1, 4\} = \emptyset$  and then  $[X_5, X_2] \in \mathbb{C}Z_3$ , giving the contradiction  $[X_5, X_2] = 0$ .

In case b)  $\lambda_1 \lambda_2 \lambda_3 = 1$ , and therefore  $\lambda_4 \lambda_5 = 1$ . Thus  $[X_5, X_4] = 0$ , and we may assume that  $0 \neq [X_4, X_1] \in \mathbb{C}Z_3$  and  $0 \neq [X_5, X_2] \in \mathbb{C}Z_2$ . Therefore,  $\lambda_5 \lambda_2 = \lambda_1 \lambda_3$  and  $\lambda_4 \lambda_1 = \lambda_2 \lambda_3$ , and since  $\lambda_4 \lambda_5 = 1$ , we get to the contradiction  $\lambda_3 = 1$ .

We can assume then that  $\mathfrak{n}$  satisfies condition (19) and thus without any loss of generality we can suppose that

$$(22) \quad [X_1, X_2] = Z_1 \quad [X_3, X_4] = Z_2.$$

Note that we can not have  $[X_5, X_j] = Z_3$  because this would imply  $\lambda_j = 1$  by using that  $\lambda_1 \dots \lambda_5 = 1$ . Let us say then that  $[X_5, X_j] = aZ_1$ ,  $a \neq 0$ . From (18) we have that  $j \neq 1, 2$ , and since both cases  $j = 3$  and  $j = 4$  are completely analogous, we will just analyze the case  $j = 3$ . This is

$$[X_1, X_2] = Z_1, \quad [X_3, X_4] = Z_2, \quad [X_5, X_3] = aZ_1.$$

Also, since  $Z_3 \in [\mathfrak{n}, \mathfrak{n}]$  there is  $1 \leq k, k' \leq 4$  such that  $[X_k, X_{k'}] = Z_3$ , and by the above observations, it is easy to see that

$$\{k, k'\} = \begin{cases} \{1, 3\} \text{ or (equivalently) } \{2, 3\} \\ \{1, 4\} \text{ or (equivalently) } \{2, 4\} \end{cases}$$

To finish the proof, we will see that both cases leads to a contradiction. The idea is to show that one of the  $\lambda_i$  is equal to one of the  $\mu_j$ , and since the conjugated numbers are uniquely determined, this implies that every  $\mu_j$  appears as a  $\lambda_k$ . From here it is easy to check in both cases that this is not possible.

Indeed, if  $[X_1, X_3] = Z_3$ , since  $1 = \lambda_5 \lambda_3 \lambda_3 \lambda_4 \lambda_1 \lambda_3$ , we have that  $\lambda_3^2 = \lambda_2$ . Therefore,  $\lambda_5 \lambda_3 = \lambda_1 \lambda_2 = \lambda_1 \lambda_3^2$  and so  $\lambda_5 = \lambda_1 \lambda_3 = \mu_3$ . Hence, there exists  $i$  such that  $\mu_1 = \lambda_1 \lambda_3^2 = \lambda_i$ . It is clear that  $i \neq 1, 2, 3, 5$  and if  $\lambda_1 \lambda_3^2 = \lambda_4$ , since  $1 = \lambda_1 \lambda_2 \lambda_3 \lambda_4 \lambda_1 \lambda_3 = \lambda_1^2 \lambda_3^4 \lambda_4$ , then  $1 = \lambda_1^3 \lambda_3^6 = \mu_1^3$  contradicting the fact that  $A_2$  is hyperbolic.

Now, if  $[X_1, X_4] = Z_3$ , then

- (i)  $1 = \lambda_5 \lambda_3 \lambda_3 \lambda_4 \lambda_1 \lambda_4$ , and from there  $\lambda_2 = \lambda_3 \lambda_4 = \mu_2$ , and
- (ii)  $1 = \lambda_1 \lambda_2 \lambda_3 \lambda_3 \lambda_4 \lambda_1 \lambda_4$ , hence  $\lambda_5 = \lambda_1 \lambda_4 = \mu_3$ .

Therefore, as we have observed before, there is  $k$  such that  $\mu_1 = \lambda_k$ . This implies that  $\lambda_1 \lambda_2 = \lambda_5 \lambda_3 = \lambda_k$  for some  $1 \leq k \leq 5$ . Again, it is clear that  $k \neq 1, 2, 3, 5$ , and if  $\lambda_1 \lambda_2 = \lambda_5 \lambda_3 = \lambda_4$ , then by (ii)  $\lambda_1 \lambda_4 \lambda_3 = \lambda_5 \lambda_3 = \lambda_4$  and hence  $\lambda_1 \lambda_3 = 1$ . From this, using that  $1 = \det A_2 = \lambda_4 \lambda_2 \lambda_5$ , we obtain that  $\lambda_1 \lambda_2 = \lambda_5 \lambda_3 = \frac{1}{\lambda_2 \lambda_4} \cdot \frac{1}{\lambda_1}$ , or equivalently  $\lambda_4^2 = (\lambda_1 \lambda_2)^2 = \frac{1}{\lambda_4}$  and then  $\lambda_4 = 1$  contradicting the fact that  $A_1$  is hyperbolic, and concluding the proof of case (5, 3).

**Case (6, 2).** We will prove in this case that there is, up to isomorphism, only one Anosov Lie algebra with no abelian factor. As usual, let  $A$  be an Anosov automorphism of  $\mathfrak{n}$  and  $\{X_1, \dots, X_6, Z_1, Z_2\}$  a basis of  $\mathfrak{n}_{\mathbb{C}}$  of eigenvectors of  $A$ ,  $\lambda_1, \dots, \lambda_6, \mu_1, \mu_2$  the eigenvalues as above.

As we have mentioned before, since  $\mu_1 \neq \mu_2$ , for all  $i, j$  there exists  $k$  such that  $[X_i, X_j] \in \mathbb{C}Z_k$ . Also, if  $\dim[X_i, (\mathfrak{n}_1)_{\mathbb{C}}] = 1$  for any  $i$ , then  $\mathfrak{n}_{\mathbb{C}}$  is either isomorphic to  $(\mathfrak{h}_3 \oplus \mathbb{R} \oplus \mathfrak{h}_3 \oplus \mathbb{R})_{\mathbb{C}}$  or  $(\mathfrak{h}_3 \oplus \mathfrak{h}_5)_{\mathbb{C}}$ . The first one has two real forms:  $\mathfrak{h}_3 \oplus \mathfrak{h}_3 \oplus \mathbb{R}^2$  and  $(\mathfrak{n}_{-1}^{\mathbb{Q}} \otimes \mathbb{R}) \oplus \mathbb{R}^2$ , of which only  $\mathfrak{h}_3 \oplus \mathfrak{h}_3 \oplus \mathbb{R}^2$  is Anosov by Theorem 3.6 and the classification in dimension 6. The only real form of  $(\mathfrak{h}_3 \oplus \mathfrak{h}_5)_{\mathbb{C}}$  is  $\mathfrak{h}_3 \oplus \mathfrak{h}_5$  (see Remark 2.11), and by Proposition 2.10,  $\mathfrak{h}_3 \oplus \mathfrak{h}_5$  has only one rational form with Pfaffian form  $f(x, y) = xy^2$ . It then follows from Proposition 3.5, (ii) that it is not Anosov.

Therefore, we can assume that

$$(23) \quad [X_1, X_2] = Z_1, \quad [X_1, X_3] = Z_2.$$

From this, one has that

$$(24) \quad \lambda_1^2 \lambda_2 \lambda_3 = 1, \quad \text{or equivalently} \quad \lambda_1 = \lambda_4 \lambda_5 \lambda_6.$$

In what follows, we will first show that there exist a reordering  $\beta$  of  $\{X_1, \dots, X_6\}$ , such that

$$(25) \quad [A_1]_\beta = \begin{bmatrix} \lambda & & & & & \\ & \lambda^{-1} & & & & \\ & & \nu & & & \\ & & & \nu^{-1} & & \\ & & & & \mu & \\ & & & & & \mu^{-1} \end{bmatrix},$$

and after that we will see that this implies that  $\mathfrak{n}_\mathbb{C} \simeq \mathfrak{g}_\mathbb{C}$ , the complexification of the Lie algebra defined in (4), which is proved to be Anosov in [18, Example 3.3]. Moreover,  $\mathfrak{g}$  is known to be the only real form of  $\mathfrak{g}_\mathbb{C}$  (see Remark 2.9).

To do this, let us first assume that

a)  $\lambda_i = \lambda_l$ , denoted by  $\lambda$ , for some  $1 \leq i \neq j \leq 6$ .

Thus  $\text{dgr } \lambda = 2$ , or  $\text{dgr } \lambda = 3$ , but  $\text{dgr } \lambda = 3$  is not possible. In fact, if  $\text{dgr } \lambda = 3$  then there exist a reordering of  $\{X_i\}$  such that the matrix of  $A_1$  in the new basis is

$$A_1 = \begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix}, \quad \text{where} \quad B = \begin{bmatrix} \lambda & & \\ & \mu & \\ & & (\lambda\mu)^{-1} \end{bmatrix}$$

is conjugated to an element in  $SL_3(\mathbb{Z})$ . This says that  $\lambda_1, \lambda_2, \lambda_3 \in \{\lambda, \mu, (\mu\lambda)^{-1}\}$ , and using (24) one can see that  $\lambda_1 = \lambda_2$  (or equivalently  $\lambda_1 = \lambda_3$ ), since every other choice ends up in a contradiction. Therefore, we may assume that  $\lambda_1 = \lambda_2 = \lambda$  and so  $\lambda_3 = \lambda^{-3} = \mu$ . Since every eigenvalue of  $A_1$  has multiplicity 2, we have that after a reordering if necessary,  $\lambda_4 = \lambda_5 = \lambda^2$  and  $\lambda_6 = \lambda^{-3}$ . Therefore, the matrix of  $A$  in the basis  $\beta = \{X_1, X_2, \dots, X_6, Z_1, Z_2\}$  is given by  $[A]_\beta = \begin{bmatrix} A_1 & \\ & A_2 \end{bmatrix}$  where

$$A_1 = \begin{bmatrix} \lambda & & & & & \\ & \lambda & & & & \\ & & \lambda^{-3} & & & \\ & & & \lambda^2 & & \\ & & & & \lambda^2 & \\ & & & & & \lambda^{-3} \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} \lambda^2 & \\ & \lambda^{-2} \end{bmatrix}.$$

Hence, since  $A$  is an automorphism of  $\mathfrak{n}$ , one gets that  $X_4, X_5 \in (\mathfrak{z} \cap \mathfrak{n}_1)_\mathbb{C}$ , contradicting our assumption of no abelian factor. Thus  $\text{dgr } \lambda = 2$ , from where assertion (25) easily follows.

On the other hand, if

b)  $\lambda_i \neq \lambda_j$  for all  $i \neq j$ ,

with no loss of generality, we can assume that  $[X_4, X_j] = aZ_1$ ,  $a \neq 0$ , for some  $j \in \{3, 5, 6\}$ .

If  $j = 5$  then it follows from  $1 = \det A_2 = \lambda_4 \lambda_5 \lambda_1 \lambda_3$ , that

$$(26) \quad \lambda_2 \lambda_6 = 1.$$

Now, we also have that  $[X_6, X_k] \neq 0$  for some  $k$ , and hence it is easy to see that we can either have

I)  $[X_6, X_3] = bZ_1$ ,  $b \neq 0$ , or

II)  $[X_6, X_4] = cZ_2$ ,  $c \neq 0$ , (or equivalently  $[X_6, X_5] = cZ_2$ ).

In case I),  $\lambda_6\lambda_3 = \lambda_1\lambda_2$  and so by (26) we have that  $\lambda_3 = \lambda_1\lambda_2^2$ . By using (24) we get to the contradiction  $\mu_1 = 1$ .

Concerning II), since  $\lambda_6\lambda_4 = \lambda_1\lambda_3$ , we obtain from (24) that  $\lambda_5\lambda_3 = 1$  and therefore  $\lambda_1\lambda_4 = 1$ . This together with (26) implies assertion (25). The case when  $j = 6$  is entirely analogous to the case  $j = 5$  and so we are not going to consider it.

If  $j = 3$  then  $\lambda_4\lambda_3 = \lambda_1\lambda_2$  and by (24) it is easy to see that

$$(27) \quad \lambda_3 = \lambda_5\lambda_6\lambda_2.$$

Analogously to the previous case, since  $[X_5, X_k] \neq 0$  for some  $k$ , it is easy to see that we can either have

I)  $[X_5, X_6] = bZ_1$ , or

II)  $[X_5, X_2] = cZ_2$  (or equivalently  $[X_5, X_4] = cZ_2$ ).

It is easy to deduce from the situation I) that (27) implies that  $\mu_2 = \mu_1^2$  and so both of them are equal to 1 contradicting the fact that  $A_2$  is hyperbolic.

In case II),  $\lambda_5\lambda_2 = \lambda_1\lambda_3$  and it follows from (27) that  $\lambda_6\lambda_1 = 1$ . Also, since  $\mathfrak{n}$  has no abelian factor, it is easy to see that  $[X_6, X_4] = dZ_2$ ,  $d \neq 0$ , and therefore  $\lambda_6\lambda_4 = \lambda_1\lambda_3$ . Hence, using (24) we obtain  $\lambda_2\lambda_4 = 1$ , from where assertion (25) follows.

To finish the proof we must study the case when (25) holds, that is,

$$A_1 = \begin{bmatrix} A_\lambda & & \\ & A_\nu & \\ & & A_\mu \end{bmatrix} \quad \text{where} \quad A_\eta = \begin{bmatrix} \eta & \\ & \eta^{-1} \end{bmatrix}.$$

Let  $\lambda_1 = \lambda$ ,  $\lambda_2 = \nu$  and thus, by (24),  $\lambda_3 = \frac{1}{\lambda^2\nu}$ . It is easy to see that  $\lambda_3$  is different from  $\lambda^{-1}$  or  $\nu^{-1}$ . Therefore, after a reordering if necessary, we have that

$$A_1 = \begin{bmatrix} \lambda & & & & \\ & \nu & & & \\ & & (\lambda^2\nu)^{-1} & & \\ & & & \lambda^2\nu & \\ & & & & \lambda^{-1} \\ & & & & & \nu^{-1} \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} \lambda\nu & \\ & (\lambda\nu)^{-1} \end{bmatrix}.$$

Using that  $A$  is an automorphism, one can see that  $[V_1, V_2] = 0$ , where  $V_1 = \langle X_1, X_2, X_3 \rangle_{\mathbb{C}}$  and  $V_2 = \langle X_4, X_5, X_6 \rangle_{\mathbb{C}}$ . Moreover, since  $\mathfrak{n}_{\mathbb{C}}$  has no abelian factor  $[V_1, V_2] = \langle Z_1, Z_2 \rangle_{\mathbb{C}}$ . From the classification of 2-step nilpotent Lie algebras with 2-dimensional derived algebra in terms of Pfaffian forms given in Section 2, it follows that there is only one Lie algebra satisfying these conditions and so  $\mathfrak{n}_{\mathbb{C}}$  is isomorphic to  $\mathfrak{g}_{\mathbb{C}}$ , as was to be shown.

**Case (3, 3, 2).** We will show in this case that there is no Anosov Lie algebra. We will begin by noting that since  $\mathfrak{n}_2$  has dimension three, we may assume that

$$[X_1, X_2] = Y_3, \quad [X_1, X_3] = Y_2, \quad [X_2, X_3] = Y_1,$$

where  $\{X_1, X_2, X_3\}$  and  $\{Y_1, Y_2, Y_3\}$  are basis of  $(\mathfrak{n}_1)_{\mathbb{C}}$  and  $(\mathfrak{n}_2)_{\mathbb{C}}$  of eigenvectors of  $A_1$  and  $A_2$ , respectively.

It follows that

$$(28) \quad [X_1, Y_1] = 0, \quad [X_2, Y_2] = 0, \quad [X_3, Y_3] = 0,$$

since any of them would be an eigenvector of  $A$  of eigenvalue  $\lambda_1\lambda_2\lambda_3 = 1$  and then  $A_3$  would not be hyperbolic.

On the other hand, since  $Z_1, Z_2 \in \mathfrak{n}_3$  we have that for some  $i, j, k, l$

$$[X_i, Y_j] = Z_1, \quad [X_k, Y_l] = Z_2,$$

and thus  $i \neq k$ . Indeed, if  $i = k$  then  $j \neq l$  and by (28)  $j, l \neq i$ . This would imply that  $\lambda_i \cdot \lambda_i \lambda_j \cdot \lambda_i \cdot \lambda_i \lambda_l = 1$  and so  $\lambda_i^3 = 1$ , a contradiction.

Hence we can assume that

$$[X_1, Y_j] = Z_1 \quad [X_2, Y_l] = Z_2.$$

For the pairs  $(j, l)$  we have four possibilities as follows:  $(2, 1), (2, 3), (3, 1)$  and  $(3, 3)$ . In order to discard some of them, we recall that since  $\dim \mathfrak{n}_1 = 3$ ,  $\lambda_i \neq \lambda_j$  for all  $1 \leq i, j \leq 3$  and from this, it follows that  $(j, l) \neq (3, 1)$  or  $(2, 3)$ . Indeed, if  $(j, l) = (3, 1)$  (or  $(2, 3)$ ) we have that  $\lambda_1 \lambda_1 \lambda_2 \lambda_2 \lambda_2 \lambda_3 = 1$  (or  $\lambda_1 \lambda_1 \lambda_3 \lambda_2 \lambda_1 \lambda_2 = 1$ ). Hence  $\lambda_1 \lambda_2^2 = 1$  (or  $\lambda_1^2 \lambda_2 = 1$ ) and we get to the contradiction  $\lambda_2 = \lambda_3$  (or  $\lambda_1 = \lambda_3$ ).

It is also easy to see that  $(j, l) \neq (3, 3)$  since this implies  $\lambda_1 \lambda_1 \lambda_2 \lambda_2 \lambda_1 \lambda_2$  and so  $\lambda_1 \lambda_2 = 1$ , contradicting the fact that  $A_2$  is hyperbolic. Finally, assume that  $(j, l) = (2, 1)$ , that is, in  $\mathfrak{n}_C$  we have at least the following non trivial brackets:

$$(29) \quad \begin{aligned} [X_1, X_2] &= Y_3, & [X_1, X_3] &= Y_2, & [X_2, X_3] &= Y_1, \\ [X_1, Y_2] &= Z_1, & [X_2, Y_1] &= Z_2. \end{aligned}$$

Let  $\lambda_1 = \lambda$  and  $\lambda_2 = \nu$ , then the matrix of  $A$  is given by

$$[A] = \begin{bmatrix} B & & \\ & B^{-1} & \\ & & \frac{\lambda}{\nu} \\ & & & \frac{\nu}{\lambda} \end{bmatrix}, \quad \text{where} \quad B = \begin{bmatrix} \lambda & \nu \\ & \frac{1}{\lambda\nu} \end{bmatrix},$$

and  $B$  is conjugated to an element of  $SL_3(\mathbb{Z})$ . Thus  $\frac{\lambda}{\nu}$  is an algebraic unit with  $|\frac{\lambda}{\nu}| \neq 1$  and  $\deg \frac{\lambda}{\nu} = 2$ . It is easy to see that under such conditions  $\frac{\lambda}{\nu}$  is necessarily a real number. Since the possibilities for  $\nu$  are either  $\nu = \bar{\lambda}$  or  $\frac{1}{|\lambda|^2}$ , we obtain that  $\lambda, \nu \in \mathbb{R}$ , which is a contradiction by the following lemma applied to  $\lambda^2, \nu^2$ . This concludes the proof of this case.

**Lemma 4.2.** Let  $\lambda_1, \lambda_2$  be two positive totally real algebraic integers of degree 3. If  $\lambda_1$  and  $\lambda_2$  are conjugated and units then  $\frac{\lambda_1}{\lambda_2}$  can never have degree two.

*Proof.* Let  $\lambda_1$  and  $\lambda_2$  be as in the lemma, then the minimal polynomial of  $\lambda_i$  is given by  $m_{\lambda_i}(x) = (x - \lambda_1)(x - \lambda_2)(x - \lambda_3)$ , where  $\lambda_1 \lambda_2 \lambda_3 = \pm 1$ . Since  $m_{\lambda_i}$  has its coefficients in  $\mathbb{Z}$ , we have that

$$\lambda_1 + \lambda_2 + \lambda_3 \in \mathbb{Z}, \quad \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} \in \mathbb{Z},$$

and hence

$$(30) \quad \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = d \in \mathbb{Z}.$$

On the other hand, if we assume that  $\lambda_1/\lambda_2$  has degree two then  $\frac{\lambda_1}{\lambda_2} + \frac{\lambda_2}{\lambda_1} = a \in \mathbb{Z}$ , and thus

$$\frac{\lambda_1}{\lambda_2} = \frac{a}{2} + \sqrt{\frac{a^2}{4} - 1} \quad \text{and} \quad \frac{\lambda_2}{\lambda_1} = \frac{a}{2} - \sqrt{\frac{a^2}{4} - 1}.$$

Recall that  $a \geq 2$ . We also note that  $\frac{\lambda_1}{\lambda_2} = \pm \lambda_1^2 \lambda_3$  and  $\frac{\lambda_2}{\lambda_1} = \pm \lambda_2^2 \lambda_3$ , and hence  $\lambda_1^2 = \pm \frac{1}{\lambda_3} \left( \frac{a}{2} + \sqrt{\frac{a^2}{4} - 1} \right)$  and  $\lambda_2^2 = \pm \frac{1}{\lambda_3} \left( \frac{a}{2} - \sqrt{\frac{a^2}{4} - 1} \right)$ . By replacing this in

(30) we obtain  $\pm \frac{1}{\lambda_3} a + \lambda_3^2 = d$ , or equivalently,

$$\lambda_3^3 - \lambda_3 d \pm a = 0.$$

This means that  $p(x) = x^3 - dx \pm a$  is a monic polynomial of degree 3 with coefficient in  $\mathbb{Z}$  which is annihilated by  $\lambda_3$ . Hence it is equal to the minimal polynomial of  $\lambda_3$  and then  $a = \pm 1$ , which is a contradiction since as we have observed above,  $a \geq 2$ .  $\square$

We would like to point out that in this lemma, we are strongly using the fact that  $\lambda_1$  and  $\lambda_2$  are totally real algebraic numbers and units. Indeed, if we consider  $p(x) = x^3 - 2$ , the roots of  $p$  are  $\{\lambda_1 = 2^{1/3}, \lambda_2 = \omega 2^{1/3}, \lambda_3 = \omega^2 2^{1/3}\}$ , where  $\omega^2 + \omega + 1 = 0$ . Since  $x^3 - 2$  is indecomposable over  $\mathbb{Q}$ , we have that  $\deg \lambda_i = 3$  for all  $i = 1, 2, 3$ , and however  $\lambda_2 \cdot \frac{1}{\lambda_1} = \omega$  has degree two.

**Case (4, 2, 2).** Let  $\mathfrak{n}$  be a nilpotent Lie algebra of type (4, 2, 2) and let  $A$  be an hyperbolic automorphism with eigenvectors  $\{X_1, \dots, X_4, Y_1, Y_2, Z_1, Z_2\}$ , a basis of  $\mathfrak{n}_{\mathbb{C}}$ , and corresponding eigenvalues  $\lambda_1, \dots, \lambda_4, \eta_1, \eta_2, \mu_1, \mu_2$  as in Proposition 3.4.

Since  $\eta_i = \lambda_j \lambda_k$  we have the following two possibilities:

- (I) In the decomposition of  $\eta_1 \eta_2$  as product of  $\lambda_i$  at least one of the  $\lambda_i$  appears twice, or
- (II)  $\eta_1 = \lambda_1 \lambda_2$ ,  $\eta_2 = \lambda_3 \lambda_4$ , and  $\lambda_i \neq \lambda_j$  for  $1 \leq i, j \leq 4$ .

In the first case we can either have

$$(a) \eta_1 = \lambda_1 \lambda_2, \eta_2 = \lambda_1 \lambda_3, \quad (b) \eta_1 = \lambda_1^2, \eta_2 = \lambda_2 \lambda_3, \quad \text{or} \quad (c) \eta_1 = \lambda_1^2, \eta_2 = \lambda_1^{-2}.$$

Note that (a) and (b) implies that  $\lambda_1 \lambda_2 \cdot \lambda_1 \lambda_3 = 1$  and hence  $\lambda_4 = \lambda_1$ . Thus  $\deg \lambda_4 = \deg \lambda_1 = 2$ , and moreover,  $\lambda_2 = \lambda_3 = \pm \lambda_1^{-1}$ . Therefore in case (a) we get to the contradiction  $\eta_1 = \eta_2 = \pm 1$ , and case (b) becomes (c).

So it remains to study case (c). There is no lost of generality in assuming that  $\lambda_1 = \lambda_2 = \lambda$  and  $\lambda_3 = \lambda_4 = \lambda^{-1}$  and from this, using the Jacobi identity, it is easy to see that the possible nonzero brackets are

$$(31) \quad \begin{aligned} [X_1, X_2] &= Y_1, & [X_2, Y_1] &= a Z_1 & [X_1, Y_1] &= a' Z_1 \\ [X_3, X_4] &= Y_2, & [X_3, Y_2] &= b Z_2. & [X_4, Y_2] &= b' Z_2. \end{aligned}$$

Since  $\mathfrak{n}_{\mathbb{C}}$  has no abelian factor, we have that  $a \neq 0$  or  $a' \neq 0$  and  $b \neq 0$  or  $b' \neq 0$ . Let  $\mathfrak{n}_0$  be the ideal of  $\mathfrak{n}_{\mathbb{C}}$  generated by  $\{X_1, X_2, Y_1, Z_1\}$  and  $\mathfrak{n}'_0$  the ideal generated by  $\{X_3, X_4, Y_2, Z_2\}$ . By the above observation, they are both four dimensional 3-step complex nilpotent Lie algebras. It is well known that there is up to isomorphism only one of such Lie algebras and therefore  $\mathfrak{n}_0$  and  $\mathfrak{n}'_0$  are both isomorphic to  $(\mathfrak{l}_4)_{\mathbb{C}}$  and  $\mathfrak{n}_{\mathbb{C}} = (\mathfrak{l}_4 \oplus \mathfrak{l}_4)_{\mathbb{C}}$ . By Remark 2.15, we know that  $\mathfrak{l}_4 \oplus \mathfrak{l}_4$  is the only real form of  $(\mathfrak{l}_4 \oplus \mathfrak{l}_4)_{\mathbb{C}}$ , and it is Anosov by Theorem 3.7. This concludes case (I).

We will now study case (II). We can assume that

$$(32) \quad [X_1, X_2] = Y_1, \quad [X_3, X_4] = Y_2.$$

Moreover, due to our assumption it is easy to see that there is no more non-trivial Lie brackets among them. On the other hand, we have that  $Z_i \in \mathfrak{n}_3$  and then for each  $i = 1, 2$

$$Z_i = [X_{j_i}, Y_{k_i}].$$

If  $k_1 = k_2$  we may assume that  $k_1 = k_2 = 1$ . By using Jacobi identity and the previous observation, one can see that  $j_1, j_2 \notin \{3, 4\}$ , and hence we get

$$[X_1, Y_1] = Z_1, \quad [X_2, Y_1] = Z_2.$$

From this we have that  $\lambda_1 \cdot \lambda_1 \lambda_2 \cdot \lambda_2 \cdot \lambda_1 \lambda_2 = 1$  and therefore  $\lambda_1^3 \lambda_2^3 = 1$ , a contradiction.

Otherwise, we can assume that  $k_1 = 1$  and  $k_2 = 2$ . Therefore  $\lambda_{j_1} \lambda_1 \cdot \lambda_{j_2} \lambda_2 \cdot \lambda_3 \lambda_4 = 1$  and then  $\lambda_{j_1} \lambda_{j_2} = 1$ . Hence  $j_1 \neq j_2$  and since  $\lambda_1 \lambda_2 \neq 1$  and  $\lambda_3 \lambda_4 \neq 1$  we can suppose that  $\lambda_1 \lambda_3 = 1$  and  $\lambda_2 \lambda_4 = 1$ . Without any loss of generality we can assume that

$$(33) \quad [X_1, Y_1] = Z_1 \quad \text{and} \quad [X_3, Y_2] = Z_2,$$

since by Jacobi  $[X_1, Y_2] = [X_3, Y_1] = 0$ . Note that we have obtained that the matrix of  $A$  is given by

$$[A_1] = \begin{bmatrix} \lambda & & & \\ & \nu & & \\ & & \lambda^{-1} & \\ & & & \nu^{-1} \end{bmatrix}, \quad [A_2] = \begin{bmatrix} \lambda \nu & & & \\ & (\lambda \nu)^{-1} & & \\ & & & \\ & & & \end{bmatrix} \quad \text{and} \quad [A_3] = \begin{bmatrix} \lambda^2 \nu & & & \\ & (\lambda^2 \nu)^{-1} & & \\ & & & \\ & & & \end{bmatrix}.$$

From this, since  $\lambda \neq \nu$  and  $A \in \text{Aut}(\mathfrak{n}_{\mathbb{C}})$ , it is easy to see that we can not have other nonzero Lie brackets on  $\mathfrak{n}_{\mathbb{C}}$  but (32), (33),  $[X_1, X_4] = aZ_1$  and  $[X_2, X_3] = bZ_2$ . This Lie algebra is isomorphic to the one with  $a = b = 0$  (by changing for  $\tilde{X}_4 = X_4 + Y_1$ ,  $\tilde{X}_2 = X_2 + Y_2$ ), and then  $\mathfrak{n}_{\mathbb{C}}$  is again isomorphic to  $(\mathfrak{l}_4 \oplus \mathfrak{l}_4)_{\mathbb{C}}$ .

We summarize the results obtained in this section in the following

**Theorem 4.3.** *Up to isomorphism, the real Anosov Lie algebras of dimension  $\leq 8$  are:  $\mathbb{R}^n$ ,  $n = 2, \dots, 8$ ,  $\mathfrak{h}_3 \oplus \mathfrak{h}_3$ ,  $\mathfrak{f}_3$ ,  $\mathfrak{h}_3 \oplus \mathfrak{h}_3 \oplus \mathbb{R}^2$ ,  $\mathfrak{f}_3 \oplus \mathbb{R}^2$ ,  $\mathfrak{g}$ ,  $\mathfrak{h}$ , and  $\mathfrak{l}_4 \oplus \mathfrak{l}_4$ .*

## 5. CLASSIFICATION OF RATIONAL ANOSOV LIE ALGEBRAS

In Section 4, we have found all real Lie algebras of dimension  $\leq 8$  having an Anosov rational form (see Theorem 4.3 or Table 3). On the other hand, the set of all rational forms (up to isomorphism) for each of these algebras has been determined in Section 2 (see Table 2). In this section, we shall study which of these rational Lie algebras are Anosov, obtaining in this way the classification in the rational case up to dimension 8.

**Case  $\mathfrak{f}_3$  (type (3,3)).** There is only one rational form  $\mathfrak{f}_3^{\mathbb{Q}}$  in this case which is proved to be Anosov in [4] and [5, 20].

**Case  $\mathfrak{h}_3 \oplus \mathfrak{h}_3$  (type (4,2)).** The rational forms of  $\mathfrak{h}_3 \oplus \mathfrak{h}_3$  are given by  $\{\mathfrak{n}_k^{\mathbb{Q}}\}$ ,  $k \geq 1$  square-free (see Proposition 2.7). The fact that  $\mathfrak{n}_k^{\mathbb{Q}}$  is Anosov for any  $k > 1$  has been proved in several papers (see [29, 15, 1, 20]) and it also follows from the construction given in [18] and Theorem 3.7. The Pfaffian form of  $\mathfrak{n}_1^{\mathbb{Q}}$  is  $f_1(x, y) = x^2 - y^2$ , and thus it follows from Proposition 3.5, (ii), that  $\mathfrak{n}_1^{\mathbb{Q}}$  is not Anosov.

**Case  $\mathfrak{g}$  (type (6,2)).** It is proved in Section 4 that the Lie algebra  $\mathfrak{g}$  defined in (4) is the only real Anosov Lie algebra of this type, and we have seen in Proposition 2.8 that  $\mathfrak{g}$  has only one rational form, which is then the only rational Anosov Lie algebra of this type.

**Case  $\mathfrak{h}$  (type (4, 4)).** We have seen in Section 4 that the only possible real Anosov Lie algebras of this type are the real forms of  $\mathfrak{h}_{\mathbb{C}}$ , namely,  $\mathfrak{h}$  and  $\mathfrak{n}_{-1}^{\mathbb{Q}} \otimes \mathbb{R}$ . The rational forms of  $\mathfrak{h}$  are determined in Proposition 2.13; they can be parametrized by  $\mathfrak{h}_k^{\mathbb{Q}}$  with  $k$  a square-free natural number. We know that the Pfaffian form of  $\mathfrak{h}_k^{\mathbb{Q}}$  is  $f_k(x, y, z, w) = xw + y^2 - kz^2$  and then  $Hf_k = 4k$ . By renaming the basis  $\beta$  given in (17) as  $\{X_1, \dots, X_4, Z_1, \dots, Z_4\}$ , we have that the Lie bracket of the Anosov rational form  $\mathfrak{h}^{\mathbb{Q}}$  of  $\mathfrak{h}$  defined by  $\beta$  is

$$\begin{aligned} [X_1, X_3] &= Z_1 + Z_3, & [X_2, X_3] &= Z_2 - Z_4, \\ [X_1, X_4] &= Z_2 + Z_4, & [X_2, X_4] &= (a^2 - 1)(Z_1 - Z_3). \end{aligned}$$

This implies that the maps  $J_Z$ 's of  $\mathfrak{h}^{\mathbb{Q}}$  are given by

$$J_{xZ_1+yZ_2+zZ_3+wZ_4} = \begin{bmatrix} 0 & 0 & -x-z & -y-w \\ 0 & 0 & -y+w & m(-x+z) \\ x+z & y-w & 0 & 0 \\ y+w & m(x-z) & 0 & 0 \end{bmatrix},$$

where  $m = a^2 - 1$ , and then its Pfaffian form is

$$f(x, y, z, w) = mx^2 - y^2 - mz^2 + w^2,$$

with Hessian  $Hf = 16m^2$ . We know that  $\mathfrak{h}^{\mathbb{Q}}$  has to be isomorphic to  $\mathfrak{h}_k^{\mathbb{Q}}$  for some square-free natural number  $k$ , but in that case  $f \simeq_{\mathbb{Q}} f_k$  and so we would have  $Hf = q^2 Hf_k$  for some  $q \in \mathbb{Q}^*$ . Thus  $16m^2 = q^2 k$ , which implies that  $k = 1$ . This shows that the Anosov rational forms of  $\mathfrak{h}$  defined by different integers  $a$ 's are all isomorphic to  $\mathfrak{h}_1^{\mathbb{Q}}$ . In what follows, we shall prove that the other rational forms of  $\mathfrak{h}$  (i.e.  $\mathfrak{h}_k^{\mathbb{Q}}$  for  $k > 1$ ) are Anosov as well.

Fix a square free natural number  $k > 1$ . Consider the basis  $\beta = \{X_1, \dots, Z_4\}$  of  $\mathfrak{h}_k^{\mathbb{Q}}$  given in Proposition 2.13 and set  $\mathfrak{n}_1 = \langle X_1, \dots, X_4 \rangle_{\mathbb{Q}}$  and  $\mathfrak{n}_2 = \langle Z_1, \dots, Z_4 \rangle_{\mathbb{Q}}$ . Let  $(a, b) \in \mathbb{N} \times \mathbb{N}$  any solution to the Pell equation  $x^2 - ky^2 = 1$ . Let  $A : \mathfrak{h}_k^{\mathbb{Q}} \mapsto \mathfrak{h}_k^{\mathbb{Q}}$  be the linear map defined in terms of  $\beta$  by

$$(34) \quad A_1 = A|_{\mathfrak{n}_1} = \begin{bmatrix} 0 & 0 & b & -a \\ 0 & 0 & -a & kb \\ 0 & 1 & 2n & 0 \\ 1 & 0 & 0 & 2n \end{bmatrix}, \quad A_2 = A|_{\mathfrak{n}_2} = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & -a & b & 4na \\ 0 & -bk & a & 4nbk \\ -1 & -2n & 0 & 2n^2 \end{bmatrix}.$$

It is easy to check that  $A \in \text{Aut}(\mathfrak{h}_k^{\mathbb{Q}})$  for any  $n \in \mathbb{N}$ , and since  $\det A_1 = \det A_2 = 1$  we have that  $A_1, A_2 \in \text{SL}_4(\mathbb{Z})$ , that is,  $A$  is unimodular. The characteristic polynomial of  $A_1$  is  $f(x) = (x^2 - 2nx + a - \sqrt{kb})(x^2 - 2nx + a + \sqrt{kb})$  and so its eigenvalues are

$$(35) \quad \begin{aligned} \lambda_1 &= n + \sqrt{n^2 - a + \sqrt{kb}}, & \lambda_2 &= n - \sqrt{n^2 - a + \sqrt{kb}}, \\ \mu_1 &= n + \sqrt{n^2 - a - \sqrt{kb}}, & \mu_2 &= n - \sqrt{n^2 - a - \sqrt{kb}}. \end{aligned}$$

We take  $n \in \mathbb{N}$  such that  $a + \sqrt{kb} < n^2$ . Therefore  $1 < \lambda_1$  and it follows from  $\lambda_1 \lambda_2 = a - \sqrt{kb} = \frac{1}{a + \sqrt{kb}} < 1$  that  $\lambda_2 < 1$ . Also,  $1 < \mu_1$  and  $\mu_1 \mu_2 = a + \sqrt{kb} > 1$ , and hence  $\mu_2 \neq 1$ , proving that  $A_1$  is hyperbolic. The eigenvalues of  $A_2$  are all of the form  $\lambda_i \mu_j$ . Indeed, it can be checked that the eigenvector for  $\lambda_i \mu_j$  is

$$Z = Z_1 - (a - \sqrt{kb})\mu_j Z_2 - (a + \sqrt{kb})\lambda_i Z_3 + \lambda_i \mu_j Z_4.$$

Now, the fact that  $\lambda_2 < \mu_2 < \mu_1 < \lambda_1$  implies that  $\lambda_i \mu_j \neq 1$  for all  $i, j$ , showing that  $A_2$  is also hyperbolic and hence that  $A$  is an Anosov automorphism of  $\mathfrak{h}_k^{\mathbb{Q}}$ .

The above is the most direct and shortest proof of the fact that  $\mathfrak{h}_k^{\mathbb{Q}}$  is Anosov for any square free  $k > 1$ , and it consists in just checking that  $A$  is unimodular and hyperbolic. But now we would like to show where did this  $A$  come from, which will show at the same time that  $\mathfrak{h}_k^{\mathbb{Q}}$  is not Anosov for  $k < 0$ . Since the proof of Proposition 2.13 actually shows that the set of rational forms up to isomorphism of  $\mathfrak{h}_{\mathbb{C}}$  is given by

$$\{\mathfrak{h}_k^{\mathbb{Q}} : k \text{ a nonzero square free integer number}\},$$

this will prove that the real completion  $\mathfrak{n}_{-1}^{\mathbb{Q}} \otimes \mathbb{R}$  of those with  $k < 0$  is not Anosov.

First of all, it is easy to see that any  $\tilde{A}$  of the form

$$(36) \quad \tilde{A}_1 = \tilde{A}|_{\mathfrak{n}_1} = \begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix}, \quad \tilde{A}_2 = \tilde{A}|_{\mathfrak{n}_2} = \begin{bmatrix} b_{11}C & b_{12}C \\ b_{21}C & b_{22}C \end{bmatrix} \quad (B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}),$$

where  $B, C \in GL_2(\mathbb{C})$ , is an automorphism of  $\mathfrak{h}_{\mathbb{C}}$ , for which we are considering the basis  $\alpha = \{X_1, \dots, Z_4\}$  with Lie bracket defined as in (5). Moreover, this forms a subgroup of  $\text{Aut}(\mathfrak{h}_{\mathbb{C}})$  containing the connected component at the identity, since any other automorphism restricted to  $(\mathfrak{n}_1)_{\mathbb{C}}$  has the form  $\begin{bmatrix} 0 & * \\ * & 0 \end{bmatrix}$ . By taking  $\tilde{A}^2$  if necessary, we can assume that if  $\mathfrak{h}_k^{\mathbb{Q}}$  is Anosov then it admits an Anosov automorphism of the form (36). The change of basis matrix  $P_k$  from the basis  $\beta_k$  of the rational form isomorphic to  $\mathfrak{h}_k^{\mathbb{Q}}$  given in the proof of Proposition 2.13 to the basis  $\alpha$  is

$$P_k|_{\mathfrak{n}_1} = \begin{bmatrix} \sqrt{k} & 1 & 0 & 0 \\ 0 & 0 & 1 & \sqrt{k} \\ -\sqrt{k} & 1 & 0 & 0 \\ 0 & 0 & 1 & -\sqrt{k} \end{bmatrix}, \quad P_k|_{\mathfrak{n}_2} = \begin{bmatrix} 2\sqrt{k} & 0 & 0 & 0 \\ 0 & \sqrt{k} & -1 & 0 \\ 0 & \sqrt{k} & 1 & 0 \\ 0 & 0 & 0 & -2\sqrt{k} \end{bmatrix},$$

and hence

$$P_k^{-1}|_{\mathfrak{n}_1} = \frac{1}{2} \begin{bmatrix} 1/\sqrt{k} & 0 & -1/\sqrt{k} & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1/\sqrt{k} & 0 & -1/\sqrt{k} \end{bmatrix}, \quad P_k^{-1}|_{\mathfrak{n}_2} = \frac{1}{2} \begin{bmatrix} 1/\sqrt{k} & 0 & 0 & 0 \\ 0 & 1/\sqrt{k} & 1/\sqrt{k} & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1/\sqrt{k} \end{bmatrix}.$$

We then have that  $A = P_k^{-1} \tilde{A} P_k \in \text{Aut}(\mathfrak{h}_k^{\mathbb{Q}})$  if and only if  $A_1 = P_k^{-1} \tilde{A}_1 P_k$  and  $A_2 = P_k^{-1} \tilde{A}_2 P_k$  belong to  $GL_4(\mathbb{Q})$ . A straightforward computation shows that  $A_1, A_2 \in GL_4(\mathbb{Z})$  (i.e.  $A$  is unimodular) if and only if  $B \in GL_2(\mathbb{Z}[\sqrt{k}])$ ,  $C = \overline{B}$  and  $\det B \det \overline{B} = \pm 1$ . Here,  $\mathbb{Z}[\sqrt{k}]$  is the integer ring of the quadratic numberfield  $\mathbb{Q}[\sqrt{k}]$  and the conjugation is defined, as usual, by  $x + \sqrt{k}b = x - \sqrt{k}b$  for all  $x, y \in \mathbb{Q}$ . Recall that if  $\det B = a - \sqrt{k}b$ ,  $a, b \in \mathbb{Z}$ , and we assume that  $\det B \det \overline{B} = 1$ , then  $a^2 - kb^2 = 1$ , the Pell equation. In order to make easier the computation of eigenvalues we can take  $B$  in its rational form, say

$$B = \begin{bmatrix} 0 & -a + \sqrt{k}b \\ 1 & 2n \end{bmatrix}, \quad \overline{B} = \begin{bmatrix} 0 & -a - \sqrt{k}b \\ 1 & 2n \end{bmatrix},$$

for some  $n \in \mathbb{Z}$ . This implies that the characteristic polynomial of  $\tilde{A}_1$  is  $f(x) = (x^2 - 2nx + a - \sqrt{k}b)(x^2 - 2nx + a + \sqrt{k}b)$  and so the eigenvalues of  $\tilde{A}_1$  and  $A_1$  are as in (35). Concerning the hyperbolicity, if  $k < 0$  then either  $b = 0$  or  $a = 0$  and  $k = -1$ , which in any case implies that  $|\mu_1 \mu_2| = 1$ , a contradiction. Therefore  $\mathfrak{h}_k^{\mathbb{Q}}$  is not Anosov for  $k < 0$ , as was to be shown. For  $k > 0$ , we can easily see that conditions  $a, b, n \in \mathbb{N}$ ,  $a + \sqrt{k}b < n^2$ , are enough for the hyperbolicity of  $A_1$ . For  $A_2$ , we can use the following general fact: the eigenvalues of a matrix of the form  $\tilde{A}_2$  in (36) are precisely the possible products between eigenvalues of  $B$  and eigenvalues of  $C$ ; and so the hyperbolicity of  $A_2$  follows as in the short proof.

<i>Real Anosov Lie algebra</i>	<i>Dimension</i>	<i>Type</i>	<i>Anosov rat. forms</i>	<i>Non – Anosov rat. forms</i>	<i>Signature</i>
$\mathbb{R}^n, 2 \leq n \leq 8$	$n$	$n$	$\mathbb{Q}^n$	--	any
$\mathfrak{h}_3 \oplus \mathfrak{h}_3$	6	(4, 2)	$\mathfrak{n}_k^{\mathbb{Q}}, k > 1$	$\mathfrak{n}_1^{\mathbb{Q}}$	{3, 3}
$\mathfrak{f}_3$	6	(3, 3)	$\mathfrak{f}_3^{\mathbb{Q}}$	--	{3, 3}
$\mathfrak{h}_3 \oplus \mathfrak{h}_3 \oplus \mathbb{R}^2$	8	(6, 2)	$\mathfrak{n}_k^{\mathbb{Q}} \oplus \mathbb{Q}^2, k > 1$	$\mathfrak{n}_1^{\mathbb{Q}} \oplus \mathbb{Q}^2$	{4, 4}
$\mathfrak{f}_3 \oplus \mathbb{R}^2$	8	(5, 3)	$\mathfrak{f}_3^{\mathbb{Q}} \oplus \mathbb{Q}^2$	--	{4, 4}
$\mathfrak{g}$	8	(6, 2)	$\mathfrak{g}^{\mathbb{Q}}$	--	{4, 4}
$\mathfrak{h}$	8	(4, 4)	$\mathfrak{h}_k^{\mathbb{Q}}, k \geq 1$	--	{4, 4}
$\mathfrak{l}_4 \oplus \mathfrak{l}_4$	8	(4, 2, 2)	$\mathfrak{l}_k^{\mathbb{Q}}, k > 1$	$\mathfrak{l}_1^{\mathbb{Q}}$	{4, 4}

TABLE 3. Real and rational Anosov Lie algebras of dimension  $\leq 8$ .

We finally note that  $A = P_k^{-1} \tilde{A} P_k$  with this  $B$  is precisely the automorphism proposed in (34).

**Remark 5.1.** An alternative proof of the fact that any rational form of  $\mathfrak{h}$  is Anosov can be given by using [4, Corollary 2.3]. Indeed, the subgroup

$$S = SL_2(\mathbb{R}) \times SL_2(\mathbb{R}) = \{A \in \text{Aut}(\mathfrak{h}) : A_1 = \begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix}, \quad B, C \in SL_2(\mathbb{R})\}$$

is connected, semisimple and all its weights on  $\mathfrak{h}$  are non-trivial. Recall that such a corollary can not be applied to the cases  $\mathfrak{h}_3 \oplus \mathfrak{h}_3$  and  $\mathfrak{l}_4 \oplus \mathfrak{l}_4$ , as they admit a rational form which is not Anosov.

**Case  $\mathfrak{l}_4 \oplus \mathfrak{l}_4$  (type (4, 2, 2)).** The rational forms of  $\mathfrak{l}_4 \oplus \mathfrak{l}_4$  are determined in Proposition 2.14 and they are denoted by  $\mathfrak{l}_k^{\mathbb{Q}}$ ,  $k$  a square free natural number. Let  $\beta$  denote the basis of  $\mathfrak{l}_k^{\mathbb{Q}}$  given in the proposition. For  $a \in \mathbb{Z}$ ,  $a \geq 2$ , consider the hyperbolic matrix

$$B = \begin{bmatrix} a & a^2 - 1 \\ 1 & a \end{bmatrix} \in SL_2(\mathbb{Z}),$$

with eigenvalues  $\lambda_1 = a + (a^2 - 1)^{\frac{1}{2}}$  and  $\lambda_2 = a - (a^2 - 1)^{\frac{1}{2}}$ . It is easy to check that the linear map  $A : \mathfrak{l}_b^{\mathbb{Q}} \longrightarrow \mathfrak{l}_b^{\mathbb{Q}}$  whose matrix in terms of  $\beta$  is

$$[A]_{\beta} = \begin{bmatrix} B & & \\ & \ddots & \\ & & B \end{bmatrix}$$

is an automorphism of  $\mathfrak{l}_b^{\mathbb{Q}}$  for  $b = a^2 - 1$ .  $A$  is hyperbolic since  $\lambda_1 > 1 > \lambda_2$  and it is unimodular by definition, so that  $A$  is an Anosov automorphism. Recall that  $\mathfrak{l}_k^{\mathbb{Q}} \simeq \mathfrak{l}_{k'}^{\mathbb{Q}}$  if and only if  $k = q^2 k'$  for some  $q \in \mathbb{Q}^*$  (see Proposition 2.14). Given a square-free natural number  $k > 1$ , there always exist  $a, q \in \mathbb{Z}$  such that  $a^2 - 1 = q^2 k$  (Pell equation), and thus any  $\mathfrak{l}_k^{\mathbb{Q}}$  with  $k > 1$  square free is Anosov.

We now prove that  $\mathfrak{l}_1^{\mathbb{Q}}$  is not Anosov. In the proof of Proposition 2.14 we have showed that any  $A \in \text{Aut}(\mathfrak{l}_1^{\mathbb{Q}})$  has the form (7) and satisfies

$$qf(z, w) = f(A_4^t(z, w)) \quad \forall (z, w) \in \mathbb{Q}^2,$$

where  $q = \det A_3 A_1$  and  $f(z, w) = z^2 - w^2$ . In the same spirit of Proposition 3.5, this implies that  $A_4^t$  leaves a finite set invariant and so it can never be hyperbolic.

The results obtained in this section can be summarized as follows.

**Theorem 5.2.** *Up to isomorphism, the rational Anosov Lie algebras of dimension  $\leq 8$  are*

- $\mathbb{Q}^n$ ,  $n = 2, \dots, 8$ ,  $(\mathbb{R}^n)$ ,
- $\mathfrak{n}_k^{\mathbb{Q}}$ ,  $k \geq 2$ ,  $(\mathfrak{h}_3 \oplus \mathfrak{h}_3)$ ,
- $\mathfrak{f}_3^{\mathbb{Q}}$ ,  $(\mathfrak{f}_3)$ ,
- $\mathfrak{n}_k^{\mathbb{Q}} \oplus \mathbb{Q}^2$ ,  $k \geq 2$ ,  $(\mathfrak{h}_3 \oplus \mathfrak{h}_3 \oplus \mathbb{R}^2)$ ,
- $\mathfrak{f}_3^{\mathbb{Q}} \oplus \mathbb{Q}^2$ ,  $(\mathfrak{f}_3 \oplus \mathbb{R}^2)$ ,
- $\mathfrak{g}^{\mathbb{Q}}$ ,  $(\mathfrak{g})$ ,
- $\mathfrak{h}_k^{\mathbb{Q}}$ ,  $k \geq 1$ ,  $(\mathfrak{h})$ ,
- $\mathfrak{l}_k^{\mathbb{Q}}$ ,  $k \geq 2$ ,  $(\mathfrak{l}_4 \oplus \mathfrak{l}_4)$ ,

where  $k$  always run over square-free numbers and the Lie algebra between parenthesis is the corresponding real completion.

In the last column of Table 3 appear the signatures of the Anosov automorphisms found in each case. It follows from the proofs given in Section 4 that the eigenvalues of any Anosov automorphism always appear in pairs  $\{\lambda, \lambda^{-1}\}$  (with only one exception:  $\mathfrak{f}_3$ ), and thus there is only one possible signature for each nonabelian Anosov Lie algebra of dimension  $\leq 8$ .

**Corollary 5.3.** *Let  $N/\Gamma$  be a nilmanifold (or infranilmanifold) of dimension  $\leq 8$  which admits an Anosov diffeomorphism. Then  $N/\Gamma$  is either a torus (or a compact flat manifold) or the dimension is 6 or 8 and the signature is  $\{3, 3\}$  or  $\{4, 4\}$ , respectively.*

It is not true in general that there is only one possible signature for a given Anosov Lie algebra. For instance, it is easy to see that the free 2-step nilpotent Lie algebra on 4 generators admits Anosov automorphisms of signature  $\{4, 6\}$  and  $\{5, 5\}$ .

## 6. APPENDIX: ALGEBRAIC NUMBERS

We will give in the following a short summary of some results about algebraic numbers over  $\mathbb{Q}$  that are used throughout the classification. We are mainly following [17, Chapter V]. Note that we will omit information on numberfields since we are not going to need it.

An element  $\lambda \in \mathbb{C}$  is called *algebraic over*  $\mathbb{Q}$  if there exist a polynomial  $p(x) \in \mathbb{Q}[x]$  such that  $p(\lambda) = 0$ . It is easy to see that the set  $D$  of all such polynomials form an ideal in  $\mathbb{Q}[x]$  and since this is a principal ideal domain,  $D$  is generated by a single polynomial. This polynomial can be chosen to be monic, and in that case it is uniquely determined by  $\lambda$  and will be called the *minimal polynomial of*  $\lambda$ , denoted by  $m_\lambda(x)$ . Therefore, if we have an algebraic number  $\lambda$  then we can define *the degree of*  $\lambda$  as the degree of  $m_\lambda(x)$ . It will be denoted by  $\text{dgr } \lambda$ . The minimal polynomial  $m_\lambda(x)$  is irreducible over  $\mathbb{Q}$  and  $\lambda$  is not a double root of  $m_\lambda(x)$ .

If  $\lambda \neq \mu$  are two algebraic numbers, we say that they are *conjugated* if  $m_\lambda(\mu) = 0$ . Note that the numbers which are conjugated to  $\lambda$  are uniquely determined by  $\lambda$  and have the same degree.

An algebraic number  $\lambda$  is said to be an *algebraic integer* if there exists a monic polynomial  $p(x) \in \mathbb{Z}[x]$  such that  $p(\lambda) = 0$ . It can be seen that in this case,  $m_\lambda(x) \in \mathbb{Z}[x]$ , and moreover, these conditions are actually equivalent. An algebraic

number is called *totally real* if  $m_\lambda(x)$  has only real roots, that is,  $m_\lambda(x) = \prod_{i=1}^r (x - \lambda_i)$  with  $\lambda_i \in \mathbb{R}$ ,  $\lambda_1 = \lambda$ . If  $\lambda$  is a totally real algebraic number with  $\text{dgr } \lambda = r$ , set  $A_\lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_r \end{bmatrix}$ . The characteristic polynomial of  $A_\lambda$  is  $m_\lambda(x)$  and then the rational form of  $A_\lambda$  is given by

$$\begin{bmatrix} 0 & 0 & \dots & 0 & -a_0 \\ 1 & 0 & \dots & 0 & -a_1 \\ 0 & 1 & & 0 & -a_2 \\ \vdots & & \ddots & & \vdots \\ 0 & & & 1 & -a_{r-1} \end{bmatrix},$$

where  $m_\lambda(x) = x^r + a_{r-1}x^{r-1} + \dots + a_1x + a_0$ . If  $\lambda$  is an algebraic integer then  $a_i \in \mathbb{Z}$  for all  $i = 0, \dots, r-1$  and then this shows that  $A_\lambda$  is conjugated to an element in  $GL_r(\mathbb{Z})$ .

Conversely, if  $A = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_r \end{bmatrix}$  is conjugated to an element of  $GL_r(\mathbb{Z})$ , then if  $p_A(x)$  is the characteristic polynomial of  $A$ ,  $p_A(x) \in \mathbb{Z}[x]$ , and therefore  $\lambda_i$  is an algebraic integer for all  $i = 1, \dots, r$ . Concerning the degree of the  $\lambda_i$ 's as algebraic numbers in such a case, we can only say that  $1 \leq \text{dgr } \lambda_i \leq r$ . Moreover, if  $\lambda_i = \lambda_j$  for some  $i \neq j$ , and since  $\lambda$  is not a double root of  $m_\lambda(x)$ , we will have that  $m_{\lambda_i}^2(x) | p_A(x)$  and hence  $1 \leq 2 \text{dgr } \lambda_i \leq r$ .

If  $\lambda$  is an algebraic integer, we say that  $\lambda$  is a *unit* if  $1/\lambda$  is an algebraic integer as well. If it is so, then the constant coefficient  $a_0$  of  $m_\lambda(x)$  is  $(-1)^n$ , where  $n = \text{dgr } \lambda$ . Conversely, if  $a_0 = \pm 1$  then  $\lambda$  is a unit.

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